Drag and Diffusion Dynamics of Charmed Quarks in QGP

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Fokker-Planck dynamics study of the velocity distribution of charmed quarks and heavy 'onium states subject to a (thermal) q(uark)-g(luon) background facilitates understanding of the properties and dynamical evolution of the deconfined phase. We determine here the properties of the $t \to \infty$ distribution arising when soft scattering processes have established detailed balance for the velocity distribution.

1. Fokker-Planck Dynamics: Master equation; Landau's soft scattering approx; Langevin-Ito method
2. Drag and Diffusion: Coefficients defined, Fluctuation-Dissipation theorem, Eistein's model
3. Einstein's Fluctuation-Dissipation Theorem Generalized
4. Equilibrium Distribution 5. Conclusions: lessons, summary and outlook

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1 Fokker-Planck Dynamics

The object of interest is the one-particle phase space distribution function, $f(\vec{x}, \vec{p}, t)$, of heavy objects emerged in QGP of light quarks and gluons. $f(\vec{x}, \vec{p}, t)$ when multiplied by the 6-dimensional phase-space volume element $d^3x\, d^3p$, gives the probability of finding the object in this infinitesimal region of phase-space.

Two paths to the Fokker-Planck equation:

- Landau soft-scattering approximation to a master equation,
  such as the Boltzmann-Vlasov equation;
- Langevin equation
  describes the stochastic behavior of a object subject to dissipation and random noise.

1.1 Master Equation

Boltzmann-Vlasov master equation for $f = f(\vec{x}, \vec{p}, t)$:

$$\frac{\partial}{\partial t} f \; + \; \dot{\vec{x}} \cdot \nabla_x f + \dot{\vec{p}} \cdot \nabla_p f = \int d^3k \, [W(\vec{p} + \vec{k}, \vec{k}) f(\vec{x}, \vec{p} + \vec{k}, t) - W(\vec{p}, \vec{k}) f(\vec{x}, \vec{p}, t)] \, ; \tag{1}$$

$$\dot{\vec{x}} = \frac{d\vec{x}}{dt} = \frac{\vec{p}}{E}, \qquad \dot{\vec{p}} = \frac{d\vec{p}}{dt} = \vec{F}(\vec{x}) \, . \tag{2}$$

Our notation is applicable to both relativistic and classical mechanics, where the non-relativistic limit is obtained for $E \to m$.

The collision term has two parts:

the gain term: the transition rate $W(\vec{p}_1, \vec{k})$ represents the rate that a particle with momentum $\vec{p}_1 = \vec{p} + \vec{k}$ loses momentum $\vec{k}$ due to reactions with the background.

the loss term: due to scattering out of a particle into arbitrary other momentum.

DETAILED BALANCE DISTRIBUTION has gain and loss in balance

The transfer matrix W is strictly local, depending only on momenta of particles. Any dependence on position $\vec{x}$ arises indirectly.

1.2 Landau's soft scattering approximation

(Taylor) Expanding the gain term about $\vec{p}$ to second order in momentum change $\vec{k}$ leads to the Vlasov-Fokker-Planck equation:

$$\frac{\partial f}{\partial t} + \dot{x}_i \frac{\partial f}{\partial x_i} + \dot{p}_i \frac{\partial f}{\partial p_i} = \frac{\partial}{\partial p_i} A_i f + \frac{\partial}{\partial p_i}\frac{\partial}{\partial p_j} B_{ij} f \, . \tag{3}$$

The transport coefficients of drag and diffusion are, respectively the first and second moment of the master equation transfer matrix:

$$A_i = A_i(\vec{p}) = \int d^3k \, k_i \, W(\vec{p}, \vec{k}) \, , \qquad B_{ij} = B_{ij}(\vec{p}) = \frac{1}{2} \int d^3k \, k_i k_j \, W(\vec{p}, \vec{k}) \, . \tag{4}$$

Any spatial dependence is arising due to implicit dependence of W on local conditions.

It is generally believed that the Fokker-Planck equation describes well the approach to (thermal) equilibrium. However, since we shall find in realistic cases non-standard distributions, we reconfirm the applicability of this equation by presenting yet another independent and generally well studied way to motivate the form of the Fokker-Planck Eq. (3).

1.3 Ito-Langevin method

Consider the Langevin system of equations:

$$\frac{d\vec{x}}{dt} = \frac{\vec{p}}{E}\,, \qquad\qquad \frac{dp_i}{dt} = F_i(\vec{x}) + G_i(\vec{x},\vec{p}) + D_{ij}(\vec{x},\vec{p})\eta_j(t)\,, \tag{5}$$

where the noise term $\vec{\eta}$ is Gaussian white noise with $\langle \eta_i(t) \rangle = 0$, and $\langle \eta_i(t)\eta_j(t') \rangle = \delta_{ij}\,\delta(t-t')$.

Using Ito's formula one shows that this Langevin system corresponds to the Fokker-Planck equation having the form[*see e.g.* C.W. Gardiner, *Handbook of Stochastic Methods for Physics, Chemistry and the Natural Sciences*, Springer-Verlag, (Berlin, 1985)]:

$$\frac{\partial f}{\partial t} + \frac{p_i}{E}\frac{\partial f}{\partial x_i} + F_i \frac{\partial f}{\partial p_i} = \frac{-\partial G_i f}{\partial p_i} + \frac{\partial^2 (\frac{1}{2} D D^T)_{ij} f}{\partial p_i \partial p_j}\,. \tag{6}$$

Thus:

$$A_i \leftrightarrow -G_i\,, \qquad B_{ij} \leftrightarrow (\frac{1}{2} D D^T)_{ij}\,. \tag{7}$$

In the Langevin formulation the computation of D_{ij} is not obvious, as we do not have a systematic way to compute the properties of the random force from first principles.

2 Drag and Diffusion in 3-dimensions

The master equation approach offers precise definitions for the required transport coefficients. Expressing the transfer matrix element in terms of quantum reaction amplitude $\mathcal{M}$, the drag and diffusion, according to Eq. (4), are [*see e.g.* B. Svetitsky, *Phys. Rev.* D **37**, (1988); M.G. Mustafa, D. Pal, and D.K. Srivastava, *Phys. Rev.* C **57**, 889 (1998)]expressed here for the two body scattering process:

$$A_i(\vec{p}) = \frac{1}{2E_p}\int\frac{d^3k}{(2\pi)^3\,2E_k}\int\frac{d^3k'}{(2\pi)^3\,2E_{k'}}\int\frac{d^3p'}{(2\pi)^3\,2E_{p'}}\frac{1}{\gamma}\sum|\mathcal{M}|^2(2\pi)^4\delta^4(p+k-p'-k')[p_i-p_i']g(\vec{k})\tilde{g}(\vec{k}') \quad (8)$$

$$\equiv \qquad \equiv \langle\langle[p_i-p_i']\rangle\rangle\,; \quad (9)$$

$$B_{ij}(\vec{p}) \equiv \qquad \equiv \frac{1}{2}\langle\langle[(p_i-p_i')(p_j-p_j')]\rangle\rangle\,. \quad (10)$$

The heavy charm 'foreground' particle is different from the 'background' of light quarks and gluons, leading to 'linearization' of the transport equations. For each background species, there is a similar additive contribution to the reaction integral in Eq. (8). Moreover, as long as the background does not distinguish discrete quantum numbers of incoming particles (such as spin), we can combine these properties in one distribution f. In such a case, we need to average over the initial reaction channels and sum over the open final channels, akin to the evaluation of the cross section, hence the degeneracy factor γ^{-1} of the foreground. $g(\vec{k})$ is the particle density of the background and $\tilde{g}(\vec{k}) = [1 \pm g(\vec{k})]$ represents a Bose enhancement/Pauli suppression factor for the participating background particles, as appropriate. It is implied here that the background has equilibrated at the temperature $T_{\rm b}$.

2.1 Drag and Diffusion Coefficients

In a spatially homogeneous distribution, that is in absence of vectors other than $\vec{p}$, the values of A_i and B_{ij}, which depend functionally on $\vec{p}$ and the background temperature T, must be of the form:

$$
\begin{aligned}
A_i(\vec{p}, T) &= p_i A(p, T)\,, &(11)\\
B_{ij}(\vec{p}, T) &= \left(\delta_{ij} - \frac{p_i p_j}{p^2}\right) B_\perp(p, T) + \frac{p_i p_j}{p^2} B_\parallel(p, T)\,. &(12)
\end{aligned}
$$

$B_\perp(\equiv B_0)$ is the transverse and $B_\parallel(\equiv B_1)$ is the longitudinal diffusion coefficient. In terms of microscopic reaction amplitudes, these three functions are defined(see Svetitsky loc. cit.) by the following expressions:

$$
\begin{aligned}
A(p) &= \langle\langle 1 \rangle\rangle - \frac{\langle\langle \vec{p}\cdot\vec{p}\,' \rangle\rangle}{p^2}\,, &(13)\\
B_\perp(p) &= \frac{1}{4}\left[\langle\langle p'^2 \rangle\rangle - \frac{\langle\langle (\vec{p}\cdot\vec{p}\,')^2 \rangle\rangle}{p^2}\right]\,, &(14)\\
B_\parallel(p) &= \frac{1}{2}\left[\frac{\langle\langle (\vec{p}\cdot\vec{p}\,')^2 \rangle\rangle}{p^2} - 2\langle\langle \vec{p}\cdot\vec{p}\,' \rangle\rangle + p^2\langle\langle 1 \rangle\rangle\right]\,. &(15)
\end{aligned}
$$

3 Einstein's Fluctuation-Dissipation Theorem Generalized

Fokker-Planck Eq. (3)for the case of no external forces and a homogeneous background comprises a conserved current $\vec{\mathcal{P}}$ in momentum space:

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial p_i}\left(A_i f + \frac{\partial}{\partial p_j} B_{ij} f\right) \equiv -\vec{\nabla}_p \cdot \vec{\mathcal{P}}\,. \tag{16}$$

This current $\vec{\mathcal{P}}$ vanishes in presence of detailed balance distribution $f_{\rm eq}$, meaning that the gain and loss terms in the transfer matrix of the master equation balance for $f_{\rm eq}$

For the detailed balance distribution of the form:

$$f_{\rm eq}(p;T,q) = N\exp(-\Phi(p;T,q))\,. \tag{17}$$

where T, q are parameters that may be needed to characterize the shape of the distribution.

Setting $\vec{\mathcal{P}} = 0$ we find the generalized fluctuation-dissipation 'theorem':

$$A_i(\vec{p},T) = B_{ij}(\vec{p},T)\frac{\partial\Phi(\vec{p})}{\partial p_j} - \frac{\partial B_{ij}(\vec{p},T)}{\partial p_j}\,, \tag{18}$$

R. Graham and H. Haken, *Z. Physik* **243**, 289 (1971); **245**, 141 (1971); R. Graham, *Z. Physik* B, **40**, 149 (1980) provide a more general approach which classifies when such a simplification is valid, and how to extend the condition Eq. (18) when it is not valid. For our case, we have verified that Eq. (18) is valid. Even though the solution we present is very simple, and it is even just a special case of the more general considerations of Graham and Haken, it seems to have escaped attention for the interesting dynamical 3-dimensional situations.

3.1 Einstein's F-D case

The well known case considered by Einstein arises in the classical problem of a particle that travels through an ideal heat bath and undergoes linear damping (Rayleigh's particle). By symmetry AND if diffusion occurs in a one dimensional momentum space, or equivalently in 3-d, assuming that transverse and longitudinal diffusion are equal, we are setting in Eq. (18):

$$A_i = p_i \gamma \,, \qquad B_{ij} = \delta_{ij} D \,. \tag{19}$$

For the Boltzmann equilibrium distribution Eq. (17) with

$$\Phi = p^2/2mkT \,, \tag{20}$$

Eq. (18) is Einstein's well-known drag-diffusion relation for Brownian motion

$$D = kT\, m\, \gamma \,. \tag{21}$$

The 1-d case, together with Einstein's Fluctuation-Dissipation theorem for non-varying transport coefficients is the usual Fokker-Planck case of study. It allows to eliminate the need for computation of diffusion properties D, as these can than be implied from the demand that the detailed balance distribution is also Boltzmann distribution.

3.2 F-D relation in homogeneous 3-d system

Using Eqs. (11,12) and the fact that in the spatially homogeneous case the equilibrium distribution depends only on $p = |\vec{p}\,|$, Eq. (18) becomes:

$$A(p,T) = \frac{1}{p}\frac{d\Phi}{dp}B_{\|}(p,T) - \frac{1}{p}\frac{dB_{\|}}{dp} - \frac{n-1}{p^2}(B_{\|}(p,T) - B_{\perp}(p,T))\,. \tag{22}$$

When the equilibrium distribution is known *a priori* and if the diffusion coefficients are also known, then we use Eq. (22) to find the unique, consistent drag coefficient. However, the reverse is not true for $n > 1$: given the equilibrium distribution and the drag coefficient, there are two diffusion coefficients which must be simultaneously determined, which is not possible without a further assumption.

- First 'popular' unjustified method: assume that the tensor B_{ij} is diagonal, $B_{\perp} = B_{\|}$ and solve the linear first order differential equation to obtain $B_{\|}$:

$$\frac{d}{dp}\left(e^{-\Phi(p)}B_{\|}(p)\right) + pA(p)e^{-\Phi(p)} = 0\,. \tag{23}$$

- An alternative method: when $T_{\rm b}$ is the only dominant energy scale, calculate the longitudinal diffusion term $B_{\|}$ in terms of the drag and then to determine the transverse diffusion term $B_{\perp}$:

$$B_{\|} \to \frac{pA}{d\Phi/dp}\,, \qquad B_{\perp} \to B_{\|} + \frac{p}{n-1}\frac{dB_{\|}}{dp}\,. \tag{24}$$

We note that when $B_{\|} = \text{Const.}$ both recipes give the same result.

These approximate solutions are sufficient to guarantee BOLTZMANN equilibrium behavior, but they may not represent well the dynamical evolution. To be sure what is going on we need to compute all 3 (or more) transport coefficients and obtain the resulting distribution.

3.3 Stopping power is NO DRAG!

The stopping power measures the energy loss per unit distance traveled, and is equal to the energy loss per unit time divided by the particle speed. In terms of the elementary matrix elements:

$$-\frac{dE}{dx} = \langle\langle\frac{E-E'}{v}\rangle\rangle = \langle\langle\frac{E-E'}{p/E}\rangle\rangle = \langle\langle\frac{E^2-E\,E'}{p}\rangle\rangle\,. \tag{25}$$

When stopping power and drag are combined we find a relativistically invariant (scalar) quantity:

$$p_iA_i + p\frac{dE}{dx} = \langle\langle\vec{p}\cdot(\vec{p}-\vec{p}\,') - E^2 + EE'\rangle\rangle = -\frac{1}{2}\langle\langle(p_\mu - p'_\mu)^2\rangle\rangle\,. \tag{26}$$

To relate stopping power and the drag coefficient, we need to evaluate:

$$B_{00} \equiv \langle\langle(E-E')^2\rangle\rangle\,, \qquad A_0 \equiv \langle\langle E-E'\rangle\rangle\,. \tag{27}$$

The energy loss can be expressed as:

$$-\frac{dE}{dx} = \frac{p_iA_i + B_{00} - B_{ii}}{p}\,, \qquad -\frac{dE}{dt} = A_0\,. \tag{28}$$

For the Rayleigh (Einstein) particle considered earlier, noting $B_{00}\to 0$, this corresponds to an energy loss:

$$-\frac{dE}{dx} = \gamma\frac{2m}{p}\left(\frac{p^2}{2m} - \frac{3kT}{2}\right) = \gamma m(v - \frac{\langle v^2\rangle_T}{v})\,, \tag{29}$$

which vanishes precisely for (average) thermal velocity. Drag of course never vanishes. Near to the detailed balance distribution the difference between the two quantities (drag and stopping power) is most important. THEY MUST NOT BE CONFOUNDED.

4 Equilibrium Distribution

Problem: even in the simple spatially homogeneous Fokker-Planck drag or one of diffusion coefficients cannot be ‘chosen’ to ensure that the ‘correct’ equilibrium distribution. Transport coefficients must be obtained in terms of elementary reaction amplitudes, see Eqs. (13–15) We cannot expect naturally a Boltzmann distribution to arise. This is the asymptotic result with static background and including the effect of rarest LARGE MOMENTUM transfer reactions. The soft scattering limit described by FP equation which is probably the most important aspect in the rapidly evolving QGP background, will also for reason of dynamics lead to a non-Boltzmannian shape. We explore here a recently well studied class of Tsallis distributions [C. Tsallis, *J. Stat. Phys.* **52**, 479 (1988)] which depends on a temperature-like quantity T and on a parameter q:

$$\Phi_{\rm Ts} = \frac{1}{1-q} \ln\left[1-(1-q)E(p)/T\right] . \tag{30}$$

The Boltzmann distribution arises when $q \to 1$. Substituting $\Phi_{\rm Ts}$ into Eq. (22), we obtain:

$$T + (q-1)E = \frac{dE}{dp} \frac{B_\parallel}{pA + \frac{dB_\parallel}{dp} + \frac{n-1}{p}(B_\parallel - B_\perp)} . \tag{31}$$

When RHS of Eq. (31) is a constant, the Boltzmann distribution is found, and the parameter q is irrelevant. When RHS of Eq. (31) is linear in E, the Tsallis shape is the detailed balance stationary distribution.

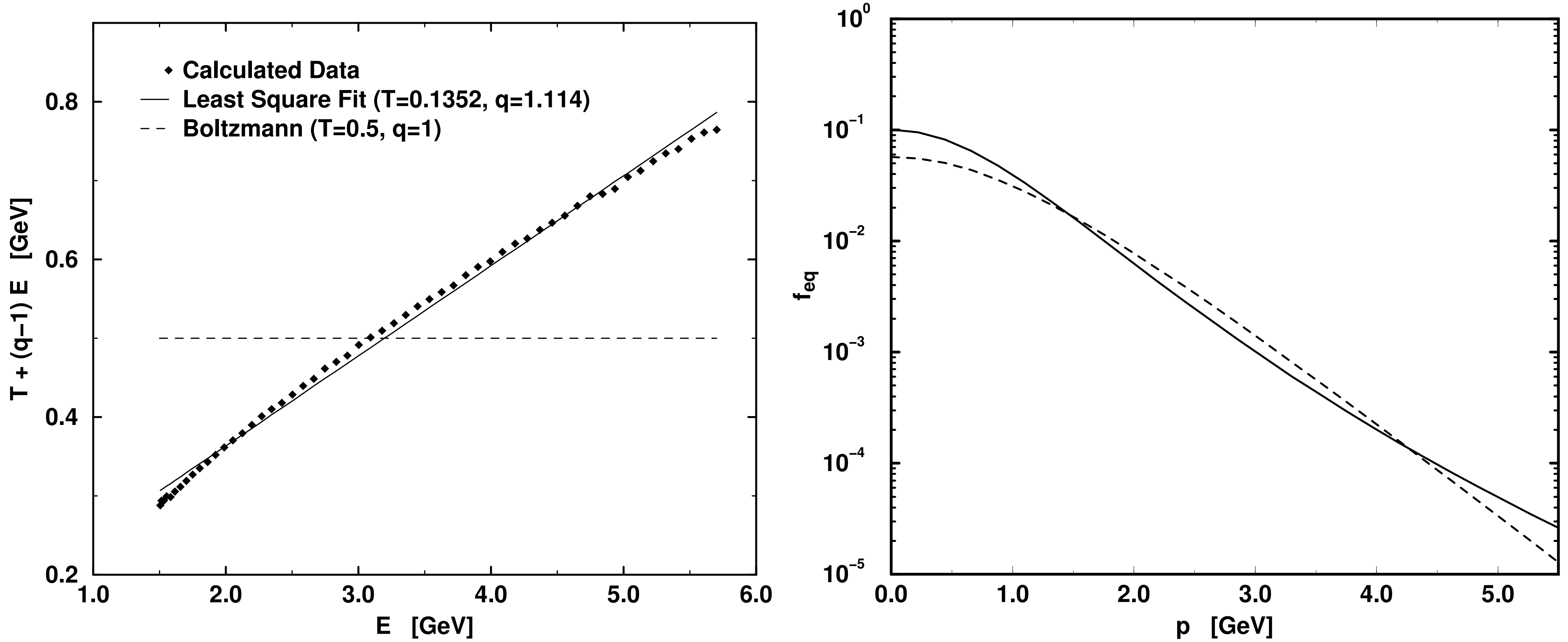
4.1 Collisional reactions in plasma

For heavy color charged particles we believe today that the bremsstrahlung process is the dominant source of drag and diffusion, while for heavy color neutral particle ('onium) we believe that soft collisional processes dominate. Unfortunately, the only presently evaluated set of transport coefficients is for collisional processes of color charged heavy quarks, and thus what follows must be seen as a first exploration of effects that should be expected.

We consider now the drag and diffusion coefficients for a charm quark with mass $m_c = 1.5\,\mathrm{GeV}$ interacting with thermal gluons at $T_\mathrm{b} = 500\,\mathrm{MeV}$ calculated using perturbative QCD techniques. We have gone to considerable length to ensure that results offered in literature apply. Diamonds in figure below show the ratio Eq. (31). The linear regression fit (straight line) shows that the parameters best describing the distribution as a Tsallis distribution are $q = 1.114$ and $T_\mathrm{T} = 135.2\,\mathrm{MeV}$. The dashed horizontal line corresponds to the Boltzmann/Jüttner distribution ($q = 1$) and $T_\mathrm{T} = T_\mathrm{b}$), which we were expecting to find. The difference to the actual distribution appears to be significant in the wide domain of charmed quark energies studied.

Tsallis distribution parameterization of the spectral shape is empirical, yet remarkably accurate, with little statistical improvement arising from further parameterization of the remaining deviation seen on LEFT. Here Calculated data (diamonds) and linear fit for the ratio in Eq. (31) for a charmed quark $m_c = 1.5\,\mathrm{GeV}$ thermalizing in gluon background at $T_\mathrm{b} = 500\,\mathrm{MeV}$. On RIGHT we show charmed quark spectrum where a solid line shows the Tsallis distribution as defined by our procedure, compared to the (dashed) expected Boltzmann/Jüttner shape for $T_\mathrm{b} = 500\,\mathrm{MeV}$ and $m_c = 1.5$ GeV. Assuming that the Tsallis shape would be measured, the spectrum would reveal two components: at low E a 'cold' Boltzmann distribution, and for high E a power-law with:

$$f_\mathrm{eq} \propto \frac{1}{E^{8.8}}\,, \qquad \mathrm{power} = \frac{1}{1-q} = -8.8\,. \tag{32}$$



5 Conclusions

5.1 Lessons

We have shown that it is relatively easy to compute the dynamics and in particular the stationary momentum distribution of heavy QCD objects in dynamically expanding QGP.

In general we will not find exact Boltzmann distributions, which could only arise if there would be a quasi stable environment in which large momentum transfer collisions had time to act; in a transient physical system only soft scattering processes equilibrate and we have seen that within collisional QCD approximation the result is a power law distribution, in fact quite similar to the behavior expected from occasional high momentum transfer collision.

Our study of the equilibrium distribution is systematic, but the results are not definitive, since the properties of transport coefficients remain today under study. Recent estimates of the bremsstrahlung stopping power in QGP [R. Baier, Yu. L. Dokshitzer, A. H. Mueller, S. Peigné and D. Schiff, *Nucl. Phys.* B **483**, 291 (1997); M.G. Mustafa, D.Pal, D.K. Srivastava and M. Thoma, *Phys. Lett.* B **428**, 234 (1998)],imply that glue radiative processes are the dominant contribution to transport coefficients of charmed quarks.

The magnitude of the drag and diffusion transport coefficients determine the magnitude of relaxation time towards equilibrium. The ratio of the transport coefficients determines the shape of the equilibrium distribution as seen in Eq. (22).

The shape of charmed quark stationary distribution is independent of the mechanisms of quark formation, is solely function of interactions with the background.

5.2 Summary and outlook

In summary, we developed tools which allow to identify within Fokker-Planck dynamics the stationary distribution for given (calculated) drag and diffusion coefficients.

We have shown that thermalization of charmed quarks in a quark-gluon plasma by collisional processes on light quarks and gluons leads to a spectral shape well parameterized by the two parameter Tsallis distribution, and we have determined the pertinent spectral parameters for the published microscopic drag/diffusion coefficients.

We are developing a schematic model allowing to understand the behavior of kinetic equilibration coefficients as function of temperature and mass. We are now studying the full dynamical description of the phase distribution of heavy QCD objects.

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