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The hadronic contributions to the anomalous magnetic moment of the muon

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Abstract

We present a new, completely revised calculation of the muon anomalous magnetic moment, $a_\mu = (g_\mu - 2)/2$, comparing it with the more recent experimental determination of this quantity; this furnishes an important test of theories of strong, weak and electromagnetic interactions. These theoretical and experimental determinations give the very precise numbers,

$$10^{11} \times a_\mu = \begin{cases} 116\,591\,806 \pm 50 \pm 10\ (\text{rad.}) \pm 30\ (\ell \times \ell) & [\text{Th., no } \tau] \\ 116\,591\,889 \pm 49 \pm 10\ (\text{rad.}) \pm 30\ (\ell \times \ell) & [\text{Theory, } \tau] \\ 116\,592\,080 \pm 60 \quad [\text{Experiment}]. \end{cases}$$

In the theoretical evaluations, the first quantity does not, and the second one does, use information from τ decay. The first errors for the theoretical evaluations include statistical plus systematic errors; the other ones are the estimated errors due to incomplete treatment of radiative corrections and the estimated error in the light-by-light scattering contribution. We thus have a significant mismatch between theory and experiment. We also use part of the theoretical calculations to give a precise evaluation of the electromagnetic coupling on the Z, $\bar{\alpha}_{\rm Q.E.D.}(M_Z^2)$, of the masses and widths of the (charged and neutral) rho resonances, of the scattering length and effective range for the P wave in $\pi\pi$ scattering, and of the quadratic radius and second coefficient of the pion form factor.

1. Introduction

The anomalous magnetic moments of electrons and muons provide one of the more impressive tests of the standard model of strong, weak and electromagnetic interactions. The electron anomaly, a_e, receives only marginal contributions from weak and strong interactions, being dominated by electromagnetic (QED) radiative corrections. The agreement between theory and experiment is such that, if we turn it around, the experimental value of a_e provides the more precise determination of the fine structure constant,[1] α.

For the muon magnetic moment, the recent precise measurements[2] of the muon anomaly, a_μ, constitute one of the more impressive tests not only of electroweak interactions, but of strong interactions as well. After these measurements, the world average value for $a_\mu = (g_\mu - 2)/2$ is

$$10^{11} \times a_\mu = 116\,592\,080 \pm 60 \quad \text{[Experiment]}. \tag{1.1}$$

Considering that theory also gives the value of the magnetic moment itself (and not only the anomaly) the agreement of theory and experiment that we will describe represents a precision of one or two parts in a *billion*. Nevertheless, and as we will see, there remains a discrepancy at the level of $2.3\,\sigma$ to $3.3\,\sigma$.

The electromagnetic and weak contributions to a_μ have been calculated with great accuracy:[3]

$$\begin{aligned} 10^{11} \times a_\mu(\text{QED}) &= 116\,584\,719 \pm 1.8, \\ 10^{11} \times a_\mu(\text{Weak}) &= \qquad\qquad 152 \pm 3. \end{aligned} \tag{1.2}$$

Combining this with (1.1), we find the *experimental* number for the hadronic contributions to a_μ,

$$10^{11} \times a_\mu(\text{Hadr.}) = 7\,209 \pm 60 \quad \text{[Experiment]}. \tag{1.3}$$

The evaluation of this quantity, $a_\mu(\text{Hadr.})$, from theory will be the main subject of the present note; we will find

$$10^{11} \times a_\mu(\text{Hadr.}) = \begin{cases} 6\,935 \pm 50 \pm 10\,(\text{rad.}) \pm 30\,(\ell \times \ell) & [\text{No } \tau] \\ 7\,018 \pm 49 \pm 10\,(\text{rad.}) \pm 30\,(\ell \times \ell) & [\text{With } \tau]. \end{cases} \tag{1.4}$$

The numbers above depend on whether or not one includes information on τ decay, which is probably the more reliable result as it is the one that incorporates more information: see our text below for details. In the errors, "rad" and "$\ell \times \ell$" refer, respectively, to estimated errors due to uncalculated radiative corrections and light-by-light scattering contributions. The results in (1.4) takes into account all the more recent e^+e^- annihilations data, and $e\pi$ scattering data. At the end of the present article we will comment on the degree of agreement of theory and experiment, and will discuss the possible reasons for the discrepancy between (1.3) and (1.4).

We will also give a summary of the results that some of the calculations imply for the electromagnetic coupling on the Z, $\bar{\alpha}_{\text{Q.E.D.}}(M_Z^2)$, for the masses and widths of the (charged and neutral) rho resonances, for the scattering length and effective radius for the P wave in pion-pion scattering, and for the quadratic radius and second coefficient of the pion form factor.

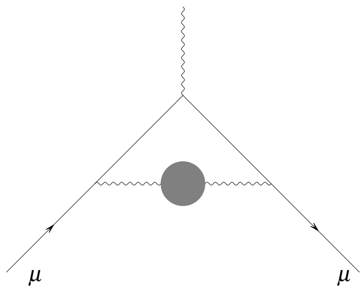


FIGURE 1. The order α^2 hadronic contributions to the muon magnetic moment. The blob represents an arbitrary hadronic state. The wavy lines are photons.

2. The hadronic contributions to a_μ. I: the $O(\alpha^2)$ piece

To order α^2, the contributions to $a_\mu(\mathrm{Hadr.})$ can be represented by the diagram shown in Fig. 1. As has been known for a long time, they can be written in terms of the cross section for e^+e^- annihilation into hadrons, as follows. We first write

$$\begin{aligned} a_\mu^{(2)}(\mathrm{Hadr.}) =& 12\pi \int_{4m_\pi^2}^\infty \mathrm{d}s\, K(s)\, \mathrm{Im}\, \Pi(s), \\ K(s) =& \frac{\alpha^2}{3\pi^2 s}\hat{K}(s); \quad \hat{K}(s) = \int_0^1 \mathrm{d}x\, \frac{x^2(1-x)}{x^2+(1-x)s/m_\mu^2}. \end{aligned} \tag{2.1a}$$

Here Π is the hadronic part of the photon vacuum polarization function. Then, we can express $\mathrm{Im}\, \Pi$ in terms of the ratio of (lowest order) cross sections for e^+e^- annihilation into hadrons over annihilation into muons:

$$R(s) = \frac{\sigma^{(0)}(e^+e^- \to \mathrm{hadrons})}{\sigma^{(0)}(e^+e^- \to \mu^+\mu^-)}, \quad \sigma^{(0)}(e^+e^- \to \mu^+\mu^-) \equiv \frac{4\pi\alpha^2}{3s}:$$

$$a_\mu^{(2)}(\mathrm{Hadr.}) = \int_{4m_\pi^2}^\infty \mathrm{d}s\, K(s) R(s). \tag{2.1b}$$

Therefore, the situation is, in principle, simple: we take the *experimental* cross section for $e^+e^- \to$ hadrons, insert it into (2.1b) and the value for $a_\mu^{(2)}(\mathrm{Hadr.})$ will follow. In practice, however, things are more complex. The experimental numbers for the cross section e^+e^- into hadrons present (relatively) large errors and we have, therefore, interest to supplement these with more precise theoretical formulas whenever possible. As a matter of fact, by so doing we are able to diminish the error in the theoretical calculation almost by a factor of two.

2.1 The low energy region, $s \leq 0.8\ \mathrm{GeV}^2$

In the region below $s = 0.8\ \mathrm{GeV}^2$, that we may call "rho region", experimental data have in fact improved substantially in the last years due to the more recent measurements, especially at Novosibirsk.[4,5] Here the important contributions are those of the omega resonance (for which experimental data and the Gounnaris–Sakurai method may be used; see below and ref. 6) and the two-pion contribution, which is the one that may be made more precise using theory, and also the one that we will discuss in more detail because it provides the bulk of $a_\mu(\mathrm{Hadr.})$. Here one profits from the fact that the two pion contribution can be expressed in terms of the pion form factor, F_π,

$$\mathrm{Im}\, \Pi_{2\pi}(s) = \frac{1}{48\pi}\left(1 - \frac{4m_\pi^2}{s}\right)^{3/2} |F_\pi(s)|^2, \tag{2.2}$$

and F_π may be determined from fits to data on $e^+e^- \to 2\pi$ and, using analyticity, also data from $\pi e \to \pi e$ scattering, i.e., at spacelike s. What is more, we may use (with due caution; see below) data on τ decay, $\tau^\pm \to \nu\pi^\pm\pi^0$, related to F_π by isospin invariance.

The properties of $F_\pi(s)$ that allow us an improved calculation are the following:

(i) $F_\pi(s)$ is an analytic function of s, with a cut from $4m_\pi^2$ to infinity.

(ii) On the cut, the phase of $F_\pi(s)$ is, because of unitarity, identical to that of the P-wave in $\pi\pi$ scattering, $\delta_1^1(s)$, and this equality holds until the opening of the inelastic threshold at $s = t_0$ (Fermi–Watson final state interaction theorem).

(iii) For large s, $F_\pi(s) \simeq 1/s$. Actually, one knows the coefficient of this behaviour, but we will not need it here.

(iv) $F_\pi(0) = 1$.

The inelastic threshold occurs, rigorously speaking, at $s = 16m_\pi^2$. However, it is an experimental fact that inelasticity is negligible until the quasi-two body channels $\omega\pi$, $a_1\pi$ ... are open. In practice one can take

$$t_0 \simeq 1\ \mathrm{GeV}^2,$$

and fix the best value for t_0 empirically. It will be $t_0 = 1.1\ \mathrm{GeV}^2$, and the dependence of our results on t_0 is very slight.

The properties (i-iv) can be taken into account with the Omnès-Muskhelishvili method.[1] We construct a function $J(s)$ with the proper phase and asymptotic behaviour by defining

$$J(s) = \mathrm{e}^{1-\delta_1^1(t_0)/\pi}\left(1 - \frac{s}{t_0}\right)^{[1-\delta_1^1(t_0)/\pi]t_0/s}\left(1 - \frac{s}{t_0}\right)^{-1}\exp\left\{\frac{s}{\pi}\int_{4m_\pi^2}^{t_0} \mathrm{d}t\, \frac{\delta_1^1(t)}{t(t-s)}\right\}. \tag{2.3a}$$

We have written the dispersion relation with one subtraction to ensure that $J(0) = 1$. The singular integral is understood to be calculated replacing $s \to s + \mathrm{i}\epsilon$, $\epsilon > 0$, $\epsilon \to 0$. We then define the function G by

$$F_\pi(s) = G(s)J(s), \tag{2.3b}$$

[1] More details about the solution of the Omnès–Muskhelishvili equations can be found in the text of N. I. Muskhelishvili, *Singular Integral Equations*, Nordhoof, 1958, and, applied to our present case, in ref. 6.

and it follows from properties (i-ii) that $G(s)$ is analytic with only the exception of a cut from t_0 to infinity, as we have already extracted the correct phase below $s = t_0$.

We can apply the effective range theory to parametrize the phase δ_1^1. According to this, the function

$$\psi(t) \equiv \frac{2k^3}{t^{1/2}} \cot \delta_1^1(t), \quad k = \frac{\sqrt{t - 4m_\pi^2}}{2} \tag{2.4a}$$

is analytic in the variable t except for two cuts: a cut from $-\infty$ to 0, and a cut from $t = t_0$ to $+\infty$. To profit from the analyticity properties of ψ we will make a conformal transformation. We define

$$w = \frac{\sqrt{t} - \sqrt{t_0 - t}}{\sqrt{t} + \sqrt{t_0 - t}}.$$

When t runs the cuts, w goes around the unit circle. We may therefore expand ψ in a power series convergent inside the unit disc. In fact, because we know that the P wave resonates (which implies a zero of ψ) it is convenient to expand not ψ itself, but the ratio $\psi(t)/(m_\rho^2 - t) \equiv \hat{\psi}(t)$. Here m_ρ is the mass of the rho resonance; so we write,

$$\psi(t) = (m_\rho^2 - t)\hat{\psi}(t) = (m_\rho^2 - t)\left\{b_0 + b_1 w + \cdots\right\}. \tag{2.4b}$$

The P-wave, $I = 1$ $\pi\pi$ scattering length, a_1^1, is related to ψ by

$$a_1^1 = \frac{1}{m_\pi \psi(4m_\pi^2)};$$

experimentally, $a_1^1 \simeq (0.038 \pm 0.003)m_\pi^{-3}$, a condition that may be incorporated into the fit. Note, however, that we do *not* assume the values of m_ρ, Γ_ρ. We only require that ψ has a zero, and will let the fits fix its location and residue. It turns out that, to reproduce the width and scattering length, and to fit the pion form factor as well, only two b_0, b_1 are needed in (2.4b).

We now turn to the function $G(s)$. This function is analytic except for a cut from $s = t_0$ to $+\infty$. The conformal transformation

$$z = \frac{\frac{1}{2}\sqrt{t_0} - \sqrt{t_0 - s}}{\frac{1}{2}\sqrt{t_0} + \sqrt{t_0 - s}}$$

maps this cut plane into the unit circle. So we may write the expansion,

$$G(s) = 1 + c_1(z + 1/3) + c_2(z^2 - 1/9) + \cdots, \tag{2.5}$$

which takes into account the condition $G(0) = 1$ order by order. We will only need two terms in the expansion, so we have c_1, c_2 as free parameters. This means that, altogether, we have the five parameters,

$$m_\rho,\ b_0,\ b_1,\ c_1,\ c_2,$$

to fit 158 experimental points.

One can then use the formulas just discussed and fit the experimental data on $F_\pi(s)$, after we have taken into account the $\omega - \rho$ interference (which includes the $\omega \to 2\pi$ piece).

This we do with the Gounnaris–Sakurai method. We write

$$F_\pi^{\rm all}(s) = F_\pi^{\rm bare}(s) \times \frac{1+\sigma \dfrac{M_\omega^2}{M_\omega^2 - s}}{1+\sigma}, \quad M_\omega = m_\omega - {\rm i}\Gamma_\omega/2 \tag{2.6a}$$

where $F_\pi^{\rm bare}$ is the form factor we would have in absence of $\omega - \rho$ interference and m_ω the (real) omega mass. We take the values of m_ω, Γ_ω from the Particle Data Tables and find

$$|\sigma| = (18 \pm 1) \times 10^{-4}, \quad \arg \sigma = 12 \pm 3^\circ\,. \tag{2.6b}$$

The fit improves when using the 2002 Novosibirsk data;[4] there is better agreement with the data of Barkov et al.,[4] in particular for the larger values of s. The parameters of the fit are, however, similar to what we found in ref. 6 using the old (1999) Novosibirsk data, as we shall see.

The result of the contribution to the hadronic part of the muon anomaly is now, fitting the e^+e^- annihilation data,[4,5]

$$10^{11} \times a_\mu^{(2)}(s \leq 0.8\,{\rm GeV}^2) = 4\,707 \pm 21; \quad \chi^2/{\rm d.o.f.} = 91/(114-7). \tag{2.7}$$

The fact that the χ^2/d.o.f. is substantially smaller than unity means that there is some room for displacement of the central value given in (2.7).

The result in (2.7) number may be improved, which we will do in two steps. First of all, we remark that, because our formulas for $F_\pi(s)$ are valid for spacelike as well as timelike s, we can use information not only from e^+e^- annihilation,[4,5] but also from πe scattering,[7] which gives $F_\pi(s)$ for negative s. We record the results of two recent evaluations, made using this method. We have our evaluation here,

$$[e^+e^-,\ \pi e]:$$

$$10^{11} \times a_\mu^{(2)}(s \leq 0.8\,{\rm GeV}^2) = \begin{cases} 4\,750 \pm 19 \quad [\text{TY; fixed norm.}]; & \dfrac{\chi^2}{\rm d.o.f.} = \dfrac{172}{159-7} \\[2ex] 4\,715 \pm 20 \pm 25 \quad [\text{TY; float. norm.}]; & \dfrac{\chi^2}{\rm d.o.f.} = \dfrac{134}{161-10}. \end{cases} \tag{2.8a}$$

The errors in the first number here do not include systematic errors in $e\pi$ scattering, which is the reason for the largish χ^2/d.o.f. In the second set, the first error is the statistical error, the second the systematic one. We take into account the systematic errors, both in e^+e^- data[4] and in the $e\pi$ data[7] as in ref. 6, by including a factor $1+\epsilon$ in the normalization of each set of data, and allowing ϵ to float. We find

$$\begin{aligned} &\epsilon_{e\pi\,[\text{ref. 7}]} = (1.3 \pm 0.2)\%, \quad \epsilon_{e^+e^-\,[\text{Akhmetshin, ref.4}]} = (-0.4 \pm 0.6)\%, \\ &\epsilon_{e^+e^-\,[\text{OLYA; Barkov, ref.4}]} = (0.5 \pm 0.9)\%; \end{aligned} \tag{2.8b}$$

the fit is depicted in Fig. 2. The only ϵ that is not compatible with zero is $\epsilon_{e\pi}$. In fact, we left its value arbitrary; that the result we found is consistent with the value quoted in ref. 7 itself, $\epsilon_{e\pi} \simeq 1\%$, is a nontrivial test of the quality of data in this reference (in particular, of their estimate of systematic errors) and of the consistency of our fitting procedure.

Then, we have a recent result by Colangelo and collaborators,[8]

$$[e^+e^-,\ \pi e]: \quad 10^{11} \times a_\mu^{(2)}(s \leq 0.8\,{\rm GeV}^2) = 4\,704 \pm 30 \quad [\text{Colangelo, ref. 8}]. \tag{2.9}$$

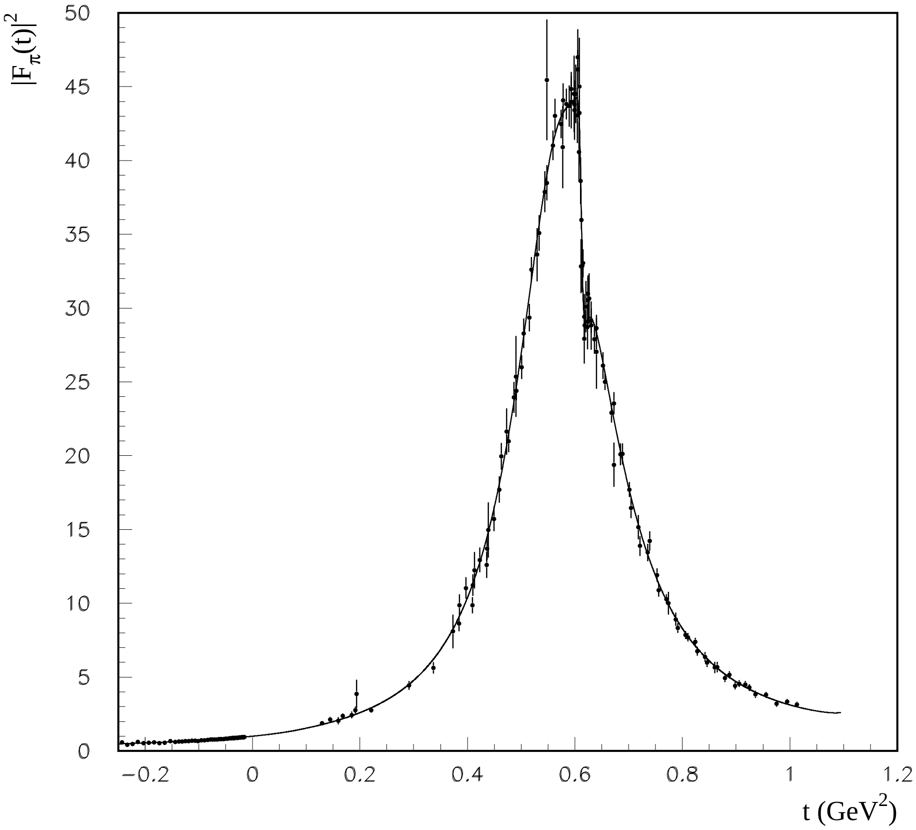


FIGURE 2. The fit to the pion form factor data in the timelike and spacelike regions; the more recent (2002) Novosibirsk data are included.

In (2.9) we have added to the Colangelo number the contribution of $\omega-\rho$ mixing, since it is apparently not included in the number reported in ref. 8. The evaluation of Colangelo and collaborators, Eq. (2.9), includes the four-pion cut, and imposes the P wave phase shift as given by chiral-dispersive evaluations.[9] This last feature, however, makes the Colangelo result vulnerable to possible defects of the chiral-dispersive evaluation, such as those discussed in ref. 10.

A way to improve the precision of (2.7) is to include information on τ decay[11] in the fit. We consider the weak vector current correlator, related to τ decay, and expand it as

$$\begin{aligned} &\Pi^V_{\mu\nu} = \mathrm{i}\int \mathrm{d}^4x\, \mathrm{e}^{\mathrm{i}p\cdot x}\langle 0|\mathrm{T}V^+_\mu(x)V_\nu(0)|0\rangle = \left(-p^2 g_{\mu\nu} + p_\mu p_\nu\right)\Pi^V(t) + p_\mu p_\nu \Pi^S(t);\\ &v_1 \equiv 2\pi\,\mathrm{Im}\,\Pi^V, \quad t = p^2. \end{aligned} \tag{2.10a}$$

We can then write

$$v_1 = \tfrac{1}{12}\left\{\left[1 - \frac{(m_{\pi^+} - m_{\pi^0})^2}{t}\right]\left[1 - \frac{(m_{\pi^+} + m_{\pi^0})^2}{t}\right]\right\}^{3/2} |F_\pi^{(\tau)}(t)|^2. \tag{2.10b}$$

To compare with the experimentally measured quantity, which involves all of $\operatorname{Im} \Pi^V_{\mu\nu}$, we have to neglect the scalar component Π^S, which is proportional to $(m_d - m_u)^2$, and thus very small. Moreover, $F_\pi^{(\tau)}$ only equals F_π in the limit of exact isospin invariance. We have also complications due to radiative corrections, that we will discuss in Subsect. 2.3.

One can take into account isospin breaking effects, at least partially, by realizing that the mass and widths of the ρ^0 and $\rho^\pm$ can be different.[6] One finds,

$$[e^+e^-,\, \pi e,\, \tau]:$$

$$10^{11} \times a_\mu^{(2)}(s \leq 0.8\,\mathrm{GeV}^2) = \begin{cases} 4\,793 \pm 17\,(\mathrm{St.}), & \dfrac{\chi^2}{\mathrm{d.o.f.}} = \dfrac{283}{241-9}; \quad [\mathrm{TY}] \\ 4\,798 \pm 17\,(\mathrm{St.}) \pm 25\,(\mathrm{Sys.}), & \dfrac{\chi^2}{\mathrm{d.o.f.}} = \dfrac{245}{244-13}; \quad [\mathrm{TY,\ Sys.}] \end{cases} \tag{2.11a}$$

depending on whether or not one takes into account systematic normalization errors. We unify the normalization of the tau decay data taking into account the relevant branching ratios, as given in the Particle Data Tables.[2]

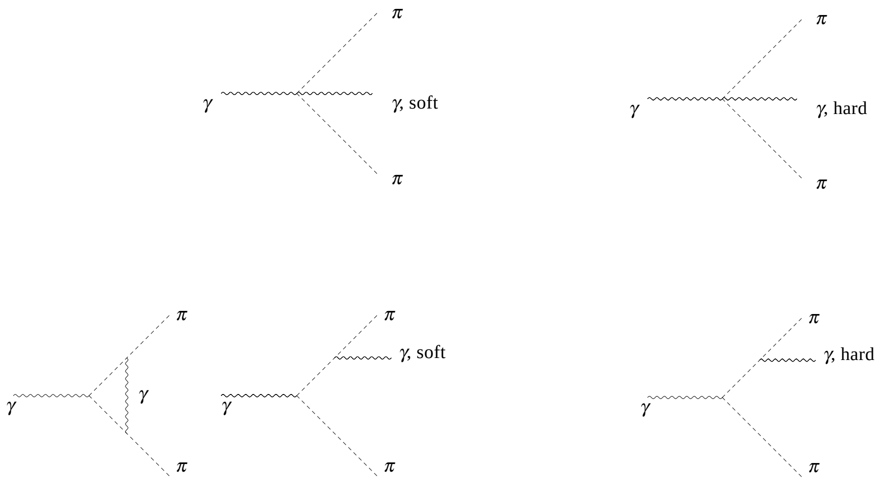
In the second set of numbers in (2.11), we have, as in (2.8), taken into account systematic errors by allowing floating normalization by a factor $1 + \epsilon$ of the various data sets. We find,

$$\begin{aligned} \epsilon_{e\pi\,[\mathrm{ref.\ 7}]} &= (1.0 \pm 0.2)\% \quad [1\%], & \epsilon_{e^+e^-\,[\mathrm{Akhmetshin,\ ref.4}]} &= (0.4 \pm 0.5)\% \quad [0.6\%], \\ \epsilon_\tau &= (-1.4 \pm 0.5)\% \quad [0.7\%], & \epsilon_{e^+e^-\,[\mathrm{OLYA;\ Barkov,\ ref.\ 4}]} &= (2.0 \pm 0.8)\% \quad [4\%]. \end{aligned} \tag{2.11b}$$

In square brackets we give the estimate of the normalization errors as given by the experimental groups themselves, except for τ decay, where the number 0.7% is taken from the Particle Data Tables. The only case where the value we find for ϵ exceeds the expectations is for τ decay data, although the difference is very small, $0.7 \pm 0.5\%$. We will discuss this again in Sect. 2.3.

The best number in (2.11a), of course, is the second, $4\,798 \pm 31$. The fact that the χ^2/d.o.f. is still a bit larger than unity can be traced partly to the size of ϵ_τ, that we discuss in Subsect. 2.3, and partly to a discrepancy between the tau decay data of OPAL and of ALEPH and CLEO. This is seen very clearly if we give the individual values for the ratio of χ^2 to number of experimental points of the various sets of data, which we do for the second

[2] When quoting the PDT we refer to the 2002 edition, K. Hagiwara et al., *Phys. Rev.* **D66**, 010001 (2002).



(A) (B)

FIGURE 3. Diagrams excluded for evaluating the pion form factor contribution in $a_\mu^{(2)}(s \leq 0.8\ \mathrm{GeV}^2)$, but included in the $O(\alpha^3)$ contribution to $a_\mu(\mathrm{Hadr.})$.

fit in (2.11a), i.e., including systematic errors:

$$\begin{array}{rc} e\pi \quad [\text{NA7, ref. 7}]: & 42/45 \\ e^+e^- \qquad [\text{ref. 4}]: & 108/113 \\ \tau\ \text{decay [Aleph, ref. 11]}: & 19/21 \\ \tau\ \text{decay [Cleo, ref. 11]}: & 32/30 \\ \tau\ \text{decay [Opal, ref. 11]}: & 40/31. \end{array} \tag{2.11c}$$

A remarkable feature of (2.11c) is that it shows that including information from tau decay in the fit does not spoil the quality of the fit of the pure e^+e^- data.

2.2. Radiative corrections for the e^+e^- case

We next devote a few words to discuss electromagnetic radiative corrections, a subject of crucial importance given the precision of the more recent data, but not always very clear in the existing literature.

The form factor that verifies the analyticity and unitary properties necessary to carry our analysis in Subsect. 2.1 is defined by

$$\langle p|J_\mu(0)|p'\rangle\Big|_{\text{electroweak int.}\equiv 0} = (2\pi)^{-3}(p-p')_\mu F_\pi((p-p')^2), \tag{2.12}$$

i.e., only strong interactions are taken into account for the expectation value of the electromagnetic current, J_μ.

This quantity has to be extracted from the experimental numbers for $e^+e^- \to \pi^+\pi^-$ cross sections, which include all sorts of radiative corrections, in particular those from photon radiation or vertex corrections, such as those in Fig. 3. As a matter of fact, these corrections are explicitly given in the 2002 version of the Novosibirsk data,[4] so in the case of $e^+e^- \to \pi^+\pi^-$ we subtract these corrections, and we can thus really get the correct quantity, defined in (2.12), to which we can apply our theoretical analysis as at the beginning of this Section. Of course, we will add the vertex corrections and both hard and soft photon radiation (Figs. 3) when discussing the $O(\alpha^3)$ corrections to $a_\mu(\mathrm{Hadr.})$. This we will do in Sect. 3.

A problem with the evaluation of the radiative corrections here, however, is that one is using a model with elementary pions, in which the form factor is included by hand (factorized). This may cause errors, whose size, likely small, we will estimate when discussing the case of tau decay in next Subsection.

2.3. Comment on combining results using $e^+e^- \to \pi\pi$ and $\tau \to \nu\pi\pi$, and on radiative corrections for τ decay

In some recent papers much ado is made about the difference in the pion form factor, in the region near and above the rho resonance, depending whether it its extracted from $e^+e^- \to \pi\pi$ or from $\tau \to \nu\pi\pi$. In fact, this difference is expected. To begin with, $|F(s)|^2$ grows almost by a factor 50 around the rho.[3] Thus, even a small difference between the masses and widths of $\rho^\pm$ and ρ^0 will imply a large difference in the form factors. This matter was studied carefully in ref. 6, which discussion we summarize and update now.

The values for the anomaly in the rho region that one obtains depending what one fits, is, with only statistical errors,

$$10^{11} \times a_\mu^{(2)}(s \leq 0.8\,\mathrm{GeV}^2) = \begin{cases} 4\,707 \pm 21 & [\text{ from } e^+e^- \to \pi\pi \text{ only}] \\ 4\,820 \pm 11 & [\text{ from } \tau \to \nu\pi\pi \text{ only}]. \end{cases} \tag{2.13}$$

For the tau, we have $\chi^2/\mathrm{d.o.f.} = 87/(83-5)$. The two numbers are well outside each other error bars and it is, therefore, dangerous to combine them in a direct manner. What, however, one can do, is to fit simultaneously $e^+e^- \to \pi\pi$ and $\tau \to \nu\pi\pi$ data allowing for different values of the parameters m_ρ, b_0, b_1, (and therefore, also different widths[4]) but with the *same* Omnès–Muskhelishvili function $G(s)$ in Eq. (2.3b). As discussed in ref. 6, we expect isospin breaking effects to be small for $G(s)$ since its imaginary part is different from zero only for $s > 1.1$ GeV^2.

Of course, when calculating $a_\mu(s \leq 0.8\,\mathrm{GeV}^2)$ one uses the parameters m_ρ, b_0, b_1 determined from the fit to $e^+e^- \to \pi\pi$, even if we use tau decay data to help fix G: we

[3] There is also a significant difference between data from e^+e^- and tau decay for 0.8 GeV$^2 < s <$ 1.1 GeV^2 which, however, affects very little the result for $a_\mu(s \leq 0.8\,\mathrm{GeV}^2)$.

[4] We take, however, equal P wave scattering lengths for $\pi^+\pi^-$ and $\pi^0\pi^\pm$. We have checked that the influence of this is negligible.

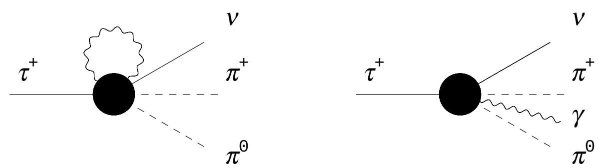


FIGURE 4. Radiative corrections to $\tau \to \nu\pi\pi$ decay.

should perhaps emphasize that the result reported in (2.11) is *not* an average of e^+e^- and τ results, but an evaluation of $e^+e^- \to \pi\pi$, using information on $G(s)$ from tau decay.

This use of different masses and widths for $\rho^\pm$, ρ^0 is not sufficient to remove the discrepancies between the form factors obtained from e^+e^- and τ decay. A reason for at least part of the remaining discrepancy is that the experimental number given in ref. 11 for τ decay includes also the radiative decay; that is to say, one does not measure the quantity $\Gamma(\tau \to \nu\pi\pi)$ but, in fact, $\Gamma(\tau \to \nu\pi\pi) + \Gamma(\tau \to \nu\pi\pi + \gamma)$ + higher orders. So we should discuss radiative corrections also for tau decay.

First of all, we must make more explicit what we used in our fits. We take the experimental numbers for the decay $\tau \to \nu\pi\pi$ including an eventual photon. The corresponding width we denote by $\Gamma_{\rm exp.}(\tau \to \nu\pi^\pm\pi^0(\gamma))$, and we divide this by the experimental decay rate for $\tau \to \nu_\tau e\nu_e$, plus eventual gammas, $\Gamma_{\rm exp.}(\tau \to \nu_\tau e\nu_e(\gamma))$. However, and as in the case of e^+e^- annihilation, we would like to have the quantity $F_\pi^{(\tau)}$ *with electroweak interactions set to zero*. Therefore, we should correct the ratio

$$\frac{\Gamma_{\rm exp.}(\tau \to \nu\pi^\pm\pi^0(\gamma))}{\Gamma_{\rm exp.}(\tau \to \nu_\tau e\nu_e(\gamma))}$$

to obtain the quantity

$$\frac{\Gamma^{(0)}(\tau \to \nu\pi^\pm\pi^0)}{\Gamma^{(0)}(\tau \to \nu_\tau e\nu_e)}$$

where the $\Gamma^{(0)}$ are evaluated to lowest order in weak interactions.

The corrections necessary to do this are as follows. We first have radiative corrections to the leptonic width, that give

$$\Gamma_{\rm exp.}(\tau \to \nu_\tau e\nu_e(\gamma)) = (1+\delta_e)\,\Gamma^{(0)}(\tau \to \nu\pi^\pm\pi^0), \quad \delta_e \simeq \left(\tfrac{25}{4} - \pi^2\right)\frac{\alpha}{2\pi} \simeq -0.004.$$

This correction was incorporated in the analysis of Subsect. 2.1.

Then we have the corrections to the hadronic width. We here have corrections similar to those in Fig. 3B; and diagrams similar to those in Fig. 3A: see Fig. 4. We start with the first, that is to say, the sum of loop corrections (plus radiation of a soft photon). As far as we know, this has not been calculated exactly. However, we may expect this to be dominated by the short distance piece, since this last contains a large logarithm, $\log M_Z/m_\tau$. This gives the correction

$$1+\frac{2\alpha}{\pi}\log\frac{M_Z}{m_\tau}\simeq 1.019 \tag{2.14}$$

as has been known from a long time.[12] This piece we have also extracted in our analysis above. After so doing, there still remains a piece of the loop correction and a correction due to soft photon radiation. Since we have extracted the large $\log M_Z/m_\tau$ piece, we expect this to be comparable to the like piece in $e^+e^-\to\pi\pi$ case, and thus small.

It remains to correct for the ratio

$$1+\delta_\gamma\equiv\frac{\Gamma(\tau\to\nu\pi\pi)+\Gamma(\tau\to\nu\pi\pi+\gamma)}{\Gamma^{(0)}(\tau\to\nu\pi\pi)}, \tag{2.15}$$

with γ a hard photon, i.e., a photon with energy larger than a given E_0. This, again, is not known, but one can approximate it by the infrared logarithmic piece, which gives an average correction

$$\delta_\gamma\simeq\frac{\alpha}{\pi}\log\frac{m_\tau-\bar{M}_{\pi\pi}}{E_0}\simeq 0.008, \tag{2.16}$$

the last for 20 MeV $\leq E_0\leq$ 80 MeV, and we have neglected terms of $O(m_\pi/m_\tau)$. $\bar{M}_{\pi\pi}$ is the average invariant mass of the two pions, that we take equal to m_ρ. Since this correction plus the remainder (after extracting the logarithm) of the loop correction are not known exactly, we have *not* included the correction (2.16) in our evaluations above: (2.16) will be part of the normalization factor ϵ_τ.

When we allowed variations of the normalization for tau decay data,[11] multiplying their numbers by $1+\epsilon$, and letting ϵ float, we found the excellent fit reported in (2.11) with $\epsilon_\tau=-1.4\%$. It unfortunately is not possible to understand all of this as due only to the neglect of radiative corrections; we have verified that, including δ_γ as given in (2.16) in the fit only changes ϵ_τ to -1.2%. We have to admit that there is a residual discrepancy with the value for the error in normalization, $\epsilon_\tau=0.7\%$, given in the Particle Data Tables; but, even with this, we still get a fit to all experimental data with a χ^2 per experimental point which is essentially unity.

In spite of this, it is clear that here we have not fully determined radiative corrections; only the logarithms $\log M_Z$, $\log E_0$ are exact in Eqs. (2.14, 16). Moreover, the function $G(s)$ will not be exactly invariant under isospin and, finally, a correction due to the fact that $m_u\neq m_d$, although likely very small, also exists beyond the rho. We must thus conclude that our partial ignorance of isospin violations, very likely dominated by radiative corrections, implies a possible shift of the central vale of the anomaly. A conservative estimate for these effects would be the difference between the two values obtained leaving the τ decay normalization fixed, and the same allowing it to float. This gives the number

$$\Delta_{\rm rad}\left[10^{11}\times a_\mu^{(2)}(s\leq 0.8\,{\rm GeV}^2)\right]\simeq 10. \tag{2.17}$$

We will accept the same error for the process $e^+e^-\to\pi\pi$, although it is probably smaller here.

2.4. The region with $s > 0.8\ \mathrm{GeV}^2$

For low energies (say, $s < 2\ \mathrm{GeV}^2$), and near quark thresholds, there is no alternative to using experimental data.[12] Between $0.8\ \mathrm{GeV}^2$ and $1.2\ \mathrm{GeV}^2$ we fit the experimental data[4] for the $\pi^+\pi^-$ channel; its contribution is $(229 \pm 3 \pm 3) \times 10^{-11}$. For other final states ($\bar{K}K$, 3π, $4\pi, \dots$) we use the e^+e^- data of refs. 5 and 13, with the methods of ref. 6. For higher energies, $s > 2\ \mathrm{GeV}^2$, and away from quark thresholds, we can use QCD formulas,[14] taking into account the more recent values of the masses of the quarks as well as the strong coupling, α_s. For the QCD calculations we take the following approximation: for n_f massless quark flavours, with charges Q_f, we write

$$R^{(0)}(s) = 3\sum_f Q_f^2 \Bigg\{ 1 + \frac{\alpha_s}{\pi} + (1.986 - 0.115 n_f)\left(\frac{\alpha_s}{\pi}\right)^2 + \left[-6.64 - 1.20 n_f - 0.005 n_f^2 - 1.240 \frac{(\sum_f Q_f)^2}{3(\sum_f Q_f^2)} \right] \left(\frac{\alpha_s}{\pi}\right)^3 \Bigg\}.$$

To this one adds mass and nonperturbative corrections. We take into account the $O(m^2)$ effect for s, c, b quarks with running masses $\bar{m}_i(s)$, which correct $R^{(0)}$ by the amount

$$-3 \sum_{i=s,c,b} Q_i^2 \bar{m}_i^2(s) \left\{ 6 + 28\frac{\alpha_s}{\pi} + (294.8 - 12.3 n_f)\left(\frac{\alpha_s}{\pi}\right)^2 \right\} s^{-1}.$$

The details may be seen in refs. 6, 15.

Adding all the contributions, one has

$$10^{11} \times a_\mu^{(2)}(s \geq 0.8\,\mathrm{GeV}^2) = 2\,134 \pm 35. \tag{2.19}$$

This number is almost the same that may be found in ref. 6, using the 1999 Novosibirsk data; including the new data has very little influence in this region.

3. The hadronic contributions to a_μ. II: the $O(\alpha^3)$ pieces

A contribution in a class by itself is the hadronic light by light one, that we label "$\ell \times \ell$". So we split

$$a(\text{Hadronic}, O(\alpha^3)) = a(\text{'One blob' hadronic}, O(\alpha^3)) + a(\ell \times \ell). \tag{3.1}$$

We will start by considering the last, given diagrammatically by graphs like that of Fig. 5. This can be evaluated only using *models*. One can make a chiral model calculation (essentially, replacing the hadronic blob by the lightest hadronic state, a π^0), or one can use a constituent quark model in which we replace the blob in Fig. 5 by a quark loop. For the chiral model calculation we have to introduce a cut-off, since the π^0 contribution diverges for large virtuality of the photon lines. The result depends on the cut-off (for the chiral calculation) or on the constituent mass chosen for the quarks. After the correction of a sign error in the evaluations of ref. 16 (see ref. 17) we find

$$10^{11} \times a(\ell \times \ell) = 86 \pm 25 \quad \text{[Chiral calculation]}. \tag{3.2a}$$

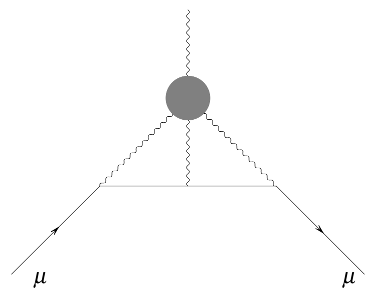


FIGURE 5. A typical diagram for the hadronic light by light contributions to the muon magnetic moment.

Earlier calculations with the π^0 model, using VMD to cure its divergence, gave (HKS, ref. 16)

$$10^{11} \times a(\ell \times \ell) = 52 \pm 20 \quad [\pi^0 \text{ pole (HKS)}]. \tag{3.2b}$$

One could also take the estimate of the π^0 pole from Hayakawa, Kinoshita and Sanda[16] *and* add the constituent quark loop, in which case we get

$$10^{11} \times a(\ell \times \ell) = 98 \pm 22 \quad [\text{Quark const. model+ pion pole}]. \tag{3.2c}$$

One expects the π^0-dominated calculation to be valid for small values of the virtual photon momenta, and the constituent model to hold for large values of the same. Thus, almost half of the contribution to $a(\ell \times \ell)$ in the chiral calculation comes from a region of momenta above 0.5 GeV, where the chiral perturbation theory starts to fail, while for this range of energies, and at least for the imaginary part of (diagonal) light by light scattering, the quark model reproduces reasonably well the experimental data, as measured in photon-photon scattering. In view of this, we will take here the figure

$$10^{11} \times a(\ell \times \ell) = 92 \pm 30, \tag{3.3}$$

but will refrain from combining this error with the others.

We next turn to the a('One blob' hadronic, $O(\alpha^3)$) corrections, which are obtained by attaching a photon or fermion loop to the various lines in Fig. 1. They can be further split into two pieces: the piece where both ends of the photon line are attached to the hadron blob, a(h.v.p., γ), *hadronic vacuum polarization* corrections, shown in Fig. 6, and the rest. So we write,

$$a(\text{'One blob' hadronic, } O(\alpha^3)) = a(\text{h.v.p., } \gamma) + a(\text{'One blob' hadronic, rest}). \tag{3.4}$$

The last can be evaluated [18] in terms of the hadronic contributions to the photon vacuum polarization, finding

$$10^{11} \times a(\text{'One blob' hadronic, rest}) = -101 \pm 6. \tag{3.5}$$

Note, however, that this result has not, as far as we know, been checked by an independent calculation.

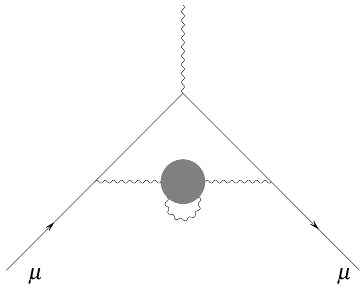


FIGURE 6. The $O(\alpha^3)$ hadronic correction $a(\text{h.v.p.},\ \gamma)$.

The only contribution that requires further discussion is that depicted in Fig. 6, $a(\text{h.v.p.},\ \gamma)$. In principle, this contribution can be evaluated straightforwardly by a generalization of the method used to evaluate the $O(\alpha^2)$ contributions. We can write

$$a^{(2)}(\text{Hadr.}) + a(\text{h.v.p.},\ \gamma) = \int_{4m_\pi^2}^{\infty} \mathrm{d}t\, K(t) R^{(2)}(t), \tag{3.6}$$

where

$$R^{(2)}(t) = \frac{\sigma^{(0)}(e^+e^- \to \text{hadrons}) + \sigma^{(2)}(e^+e^- \to \text{hadrons}) + \sigma^{(0)}(e^+e^- \to \text{hadrons};\gamma)}{\sigma^{(0)}(e^+e^- \to \mu^+\mu^-)}.$$

The notation means that we evaluate the hadron annihilation cross section to second order in α, and we add to it the first order annihilation into hadrons plus a photon. For energy large enough this can be calculated with the parton model, which leads to a (very small) correction, $(2 \pm 1) \times 10^{-11}$.

Then comes the contribution of small momenta. We start by discussing the process involving two pions. We calculate the corresponding piece by adding the contribution of the diagrams in Fig. 3, as given in the 2002 paper by Akhmetshin et al.[4] In this way, we find

$$10^{11} \times a(\text{h.v.p.},\ \pi^+\pi^-\gamma) = 47.6 \pm 0.3. \tag{3.7}$$

The number is very close to that obtained in ref. 6 (46 ± 9) but the errors have decreased drastically.[5] A similar analysis ought to be made, in principle, for other radiative intermediate states like $3\pi + \gamma$ and $\bar{K}K + \gamma$, which can be estimated in terms of the corresponding decays of the ω and ϕ, but they give a contribution below the 10^{-11} level and we neglect them. The contribution from $\pi^0\pi^0\gamma$, $(2.0 \pm 0.3) \times 10^{-11}$, is taken from ref. 7.

The lowest energy contributions to $\sigma^{(0)}(e^+e^- \to \text{hadrons};\gamma)$ are those of the intermediate states $\pi^0\gamma$ and $\eta\gamma$, Fig. 7. These contributions were evaluated in detail in ref. 6;

[5] In ref. 6 only the radiation of hard photons was included, evaluated using the results of ref. 19. The fact that this is so similar to the full result justifies the (expected) smallness of soft photon plus vertex correction.

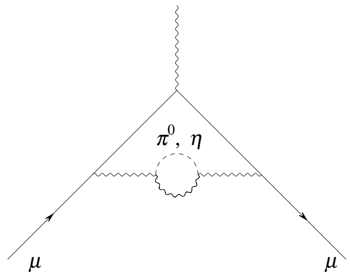


FIGURE 7. The $\pi^0\gamma$, $\eta\gamma$ contributions to a(h.v.p., γ).

we have

$$\begin{aligned} 10^{11} \times a(\text{h.v.p., } \pi^0\gamma) &= 37 \pm 3 \\ 10^{11} \times a(\text{h.v.p., } \eta\gamma) &= 6 \pm 2 \end{aligned} \tag{5.9a}$$

Collecting all of this, we get the total effect of the states $hadrons + \gamma$, including the loop correction,

$$10^{11} \times a(\text{h.v.p., } \gamma) = 95 \pm 6. \tag{3.8}$$

4. Byproducts

4.1 The electromagnetic coupling on the Z, $\bar{\alpha}_{\text{Q.E.D.}}(M_Z^2)$

With a simple change of integration kernel the previous analysis can be extended to evaluate the hadronic contribution to the QED running coupling, $\bar{\alpha}_{\text{Q.E.D.}}(t)$, in particular on the Z particle, $t = M_Z^2$; this is an important quantity that enters into precision evaluations of electroweak observables. By using a dispersion relation one can write this hadronic contribution at energy squared t as

$$\Delta_{\text{had}}\alpha(t) = -\frac{t\alpha}{3\pi}\int_{4m_\pi^2}^{\infty} \mathrm{d}s\, \frac{R(s)}{s(s-t)}$$

where R is as in (2.1b) and the integral has to be understood as a principal part integral. Therefore, we can carry over all the work from the previous sections as in ref. 20, with the simple replacement

$$K(s) \to -\frac{t\alpha}{3\pi}\frac{1}{s(s-t)}.$$

We find, to next to leading order in α,

$$10^5 \times \Delta_{\text{had}}\alpha(M_Z^2) = 2\,742 \pm 12 \tag{4.1}$$

or, excluding the top quark contribution,

$$10^5 \times \Delta_{\rm had}\alpha^{(5)}(M_Z^2) = 2\,749 \pm 12.$$

Adding the known pure QED corrections, the running QED coupling, in the momentum scheme, is

$$\bar{\alpha}_{\rm Q.E.D.}(M_Z^2) = \frac{1}{128.962 \pm 0.016}. \tag{4.2}$$

The difference with the result in ref. 20 is minute.

4.2. The masses and widths of the rho, the quadratic radius and second coefficient of the pion, and the scattering length and effective range parameter for the P wave in $\pi\pi$ scattering

As other byproducts of our analysis we can give very precise numbers for the masses and widths of the neutral and charged rho resonances, for the quadratic radius and second coefficient of the pion form factor, and for the the scattering length, a_1^1, and effective range parameter, b_1^1 (defined as in ref. 9), for the P wave in $\pi\pi$ scattering. In what regards the first, we have the following results: for the charged rho, and in MeV,

$$\begin{array}{lcc} & \text{TY},\ \tau, e\pi, e^+e^- & \text{GJ } (\tau \text{ decay only}) \\ m_{\rho^\pm} & 774.0 \pm 0.4 & 775.4 \pm 0.4 \\ \Gamma_{\rho^\pm} & 147.7 \pm 0.7 & 149.3 \pm 0.4 \\ \chi^2/\text{d.o.f.} & 1.06 & \sim 1.8. \end{array} \tag{4.3}$$

For the neutral rho,

$$\begin{array}{lccc} & \text{TY},\ e^+e^-, e\pi & \text{TY},\ \tau, e\pi, e^+e^- & \text{GJ } (e^+e^- \text{ only}) \\ m_{\rho^0} & 773.1 \pm 0.6 & 773.2 \pm 0.4 & 772.95 \pm 0.7 \\ \Gamma_{\rho^0} & 141.7 \pm 1.2 & 146.0 \pm 0.8 & 147.9 \pm 0.7 \\ \chi^2/\text{d.o.f.} & 0.9 & 1.06 & \sim 1.7 \end{array} \tag{4.4}$$

Here TY are our results here, and GJ refers to the results by Ghozzi and Jegerlehner.[21] Clearly, the masses are stable and well determined, the widths less so.

The quadratic radius and second coefficient of the pion are defined by

$$F_\pi^2(t) \underset{t\to 0}{\simeq} 1 + \tfrac{1}{6}\langle r_\pi^2\rangle t + c_\pi t^2, \tag{4.5}$$

and our results allow a precise determination of both quantities. The same is true for a_1^1, b_1^1. We have, for $\pi^+\pi^-$,

$$\begin{array}{lccc} & \text{TY [Only } e^+e^-,\ e\pi] & \text{TY [Including } \tau \text{ decay]} & \text{Colangelo} \\ \langle r_\pi^2\rangle\ \ (\text{fm}^2) & 0.423 \pm 0.003 & 0.432 \pm 0.001 & 0.435 \pm 0.005 \\ c_\pi\ (\ \text{GeV}^{-4}) & 3.78 \pm 0.05 & 3.84 \pm 0.02 & - \\ a_1^1\ (m_\pi \equiv 1) & (38.9 \pm 2.3) \times 10^{-3} & (37.8 \pm 0.8) \times 10^{-3} & (37.9 \pm 0.5) \times 10^{-3} \\ b_1^1\ (m_\pi \equiv 1) & (4.1 \pm 0.7) \times 10^{-3} & (4.74 \pm 0.09) \times 10^{-3} & (5.67 \pm 0.13) \times 10^{-3}. \end{array} \tag{4.6}$$

TY is from our results here and "Colangelo" from refs. 8, 9. The discrepancy between the result for b_1^1 from the pion form factor and that in ref. 9 had already been noted in ref. 10.

The numbers coming from our calculations for a_1^1, b_1^1, *including* tau decay data, agree much better than those not including it with results from pion-pion scattering, either from phase shifts analyses or using the Froissart–Gribov representation,[10] which is another reason for preferring the results including tau decays.

The values of the parameters of our fits are

$$\begin{aligned}&e^+e^-,\ e\pi:\\ &c_1 = 0.26 \pm 0.04, \quad c_2 = 0.19 \pm 0.13,\\ &b_0 = 1.106 \pm 0.009, \quad b_1 = 0.23 \pm 0.10\end{aligned} \tag{4.7}$$

and

$$\begin{aligned}&e^+e^-,\ e\pi,\ \tau:\\ &c_1 = 0.24 \pm 0.01, \quad c_2 = -0.18 \pm 0.03,\\ &b_0 = 1.074 \pm 0.006, \quad b_1 = 0.13 \pm 0.04.\end{aligned} \tag{4.8}$$

The numbers b_i in (4.8) correspond to $\pi^+\pi^-$; they are the ones we have used to calculate $a_\mu(s \leq 0.8\ \mathrm{GeV}^2)$. For $\pi^0\pi^+$ we would have,

$$b_0 = 1.064 \pm 0.006, \quad b_1 = 0.13 \pm 0.03 \quad [\pi^0\pi^+], \tag{4.9}$$

and the c_i are as in (4.8). The corresponding scattering length and effective range parameter are, in units of m_{π^+},

$$a_1^1 = (37.8 \pm 0.8) \times 10^{-3}, \quad b_1^1 = (4.78 \pm 0.09) \times 10^{-3} \quad [\pi^0\pi^+]. \tag{4.10}$$

5. Comparison of theoretical calculations with experiment for a_μ

We return to the magnetic moment of the muon, and present, in Table I, a summary of our results for a(Hadr.). In this Table we have added our old result from 2002, ref. 6, and the result of a recent evaluation by Jegerlehner,[21] in which theory is kept to a minimum; for example, only data are used for $s < 0.8\ \mathrm{GeV}^2$, and QCD is taken valid only for energies above 13 GeV. The main interest of this type of calculation lies in its role as control of the calculations where a more comprehensive use of theory (as well as extra experimental information) is made. In the Table we also include the recent experimental value of ref. 2. For our evaluations here, we have separated explicitly the estimated errors due to radiative corrections, and the light-by-light scattering diagram.

From Table I it is clear that there is reasonable agreement among the various theoretical determinations, but there is a definite distance between the central values from theory and experiment, at a level between $2.3\,\sigma$ and $3.3\,\sigma$, if we add quadratically "rad" and "$\ell \times \ell$" errors to the other ones. In the remaining of this section we will discuss possible reasons for this discrepancy.

An obvious reason would be new physics; but, although literally hundreds of papers have been written discussing this possibility, we find it somewhat unlikely. To get overlap of the central values of theory and experiment, the shift due to new physics should be of 190×10^{-11}, or larger. This is larger than the shift due to weak interactions (152×10^{-11}),

	$10^{11}\times a_\mu(s\leq 0.8\ \mathrm{GeV}^2)$	$10^{11}\times a_\mu(\mathrm{Hadr.})$
Jegerlehner	–	$6\,840\pm 94$
TY (2002), e^+e^-, $e\pi$, τ	$4\,774\pm 51$	$6\,993\pm 69$
TY, e^+e^-, $e\pi$	$4\,715\pm 32\pm 10\,(\mathrm{rad.})$	$6\,935\pm 50\pm 10\,(\mathrm{rad.})\pm 30\,(\ell\times\ell)$
TY, e^+e^-, $e\pi$, τ	$4\,798\pm 31\pm 10\ (\mathrm{rad.})$	$7\,018\pm 49\pm 10\,(\mathrm{rad.})\pm 30\,(\ell\times\ell)$
Experiment		$7\,209\pm 60$

TABLE I

Contributions to the rho region, and to the hadronic part of the muon anomaly. Jegerlehner: ref. 21. TY (2002): ref. 6. TY: this article.

suggesting new particles with masses similar to the W, Z masses, or smaller. However, such particles would hardly have escaped the massive searches at LEP or the Tevatron. A smaller shift due to new physics is of course not excluded by the data, but something $\lesssim 100\times 10^{-11}$ is of the order of the theoretical and/or experimental errors, hence undetectable at the present levels of accuracy.

A second reason is, of course, a displacement of the experimental result. Since the experimental number in Eq. (1.1) comes basically from only *one* experiment, it could happen that an independent determination would move it to better agreement with the results of the theoretical evaluations.

A third possibility is that the evaluation of Krause[17] contains a mistake. We will not know if this is the case until somebody checks this calculation independently. And a fourth possibility is that the central values of some of the theoretical evaluations presented here are displaced with respect to the true values. The more obvious place where such a displacement may occur is the evaluation of $a(\ell\times\ell)$. The two approximations used to evaluate this contribution do not have overlapping ranges of validity; there is a wide region, when the virtualities of some (one) of the intermediate photons are small, and at the same time other (others) are large, where neither the one-pion or the constituent quark approximations need to be valid. However, it is not easy to see how one could get the large values necessary for theory and experiment to overlap.

Finally, we may have a coincidence of several of the effects mentioned here, with the bad luck that they add.

6. Concluding remarks

Our analysis shows that, to get a precise value of a_μ, it is certainly necessary to profit from the existence of methods that allow us to make full use of theory, in particular for fitting the pion form factor: the use of robust theory in the fits produces robust results. It is not very consistent to use analyticity and unitarity to write the representation (2.1), and refuse to use exactly the same ingredients to improve the knowledge of F_π. In this sense, it is also important to take into account the data on $F_\pi(s)$ for spacelike s, i.e., from $e\pi$ scattering. This has been deemed inappropriate by some authors because they (may) contain systematic errors. However, if one refrained from taking into account data afflicted by systematic errors, there would be no data one could use. Systematic errors can and should be taken into account as shown, in this particular instance, in ref. 6 and in the present article.

In the same vein, we believe that tau decay data can and should be used, in spite of the fact that the data of the various tau decay experiments differ in some energy regions by more than one standard deviation –doubtlessly because of systematic errors, as is obvious from the table (2.11c). This is particularly important because the errors given in Eq. (2.8a) for $a_\mu^{(2)}(s \leq 0.8\ \mathrm{GeV}^2)$ are deceptively small. As we already commented, the error per experimental point for $e^+e^- \to \pi\pi$, fitting only e^+e^-, $e\pi$ data, is $89/113 \simeq 0.79$, clearly smaller than unity; while, even imposing tau decay information, the error per experimental point is (as reported in Eq. (2.11c)) only of $108/113 \simeq 0.97$, perfectly acceptable. This means that acceptable fits –like, indeed, the one obtained by us using also tau decay data– can be found outside the nominal error bars in Eq. (2.8a).[6]

Likewise, when data on F_π from the processes $e^+e^- \to \pi^+\pi^- + \gamma$ are forthcoming, they should be incorporated into the analysis: the safest way to get rid of systematic errors is to combine data of various, independent experiments, so that the various independent systematic errors average out. In view of this it is, perhaps, not surprising that the calculation reported in the last entry of TY in Table I, that includes e^+e^-, $e\pi$ and τ data, is the one closest to experiment.

It is, however, not clear to us that one can improve the results using $\pi\pi$ scattering data. If we include in the fit the *experimental* numbers for δ_1^1, the value of a_μ increases by 8×10^{-11}; but this is not necessarily more precise than the result without including this information. The systematic errors of δ_1^1, due to the fact that one does not scatter real pions (and thus one has to rely on models), are larger than the errors in our calculation. And if, like Colangelo and collaborators,[8,9] we input δ_1^1 from theoretical analyses (Roy equations and chiral perturbation theory), one is depending on determinations whose accuracy has been challenged[10] and is, very likely, too optimistically estimated.

Apart from this, to get real improvement in the theoretical predictions for $a_\mu(\mathrm{Hadr.})$ it would be, first of all, necessary to remove the sources of uncertainty mentioned at the end of Sect. 5. Of these, the one stemming from $\ell \times \ell$ is unlikely to be removed in a satisfactory manner; and we have also a problem (although less important numerically) with electroweak radiative corrections to tau decay. If we treat them by considering the pions as elementary, and factoring out F_π, then Sirlin's theorem[12] implies that, for $\tau \to \nu\pi\pi$, the logarithmic piece, $\log M_Z/m_\tau$ cancels out. It, however, does not cancel if we consider that, at short distances, the decay is really $\tau^- \to \nu\bar{u}d$. This means that the model with elementary pions fails, for this case of tau decay, and thus that the chances of removing the uncertainties due to lack of accurate knowledge of the radiative corrections here are remote.

[6] One should, however, not forget that the value we needed for the tau decay normalization error, $(-1.4 \pm 0.5)\%$ (cf. Eq. (2.11b)), or $(-1.2 \pm 0.5)\%$ if including the estimate (2.16) for δ_γ, is larger than the expected normalization error, as given by the Particle Data Tables, 0.7%.

As a final comment, we would like again to bring attention to the mismatch between the experimental and theoretical values for a_μ; although not yet definite proof of failure of the standard model, it cannot be ignored.

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