some classified as uncorrelated are correlated within a given year, but uncorrelated between years.

8 Theoretical uncertainties

It is important to assess the theoretical uncertainties associated with the BHLUMI Monte Carlo. In fact a reliable determination of the running coupling constant from Equation 2 needs a precise knowledge of the radiative corrections.

The theoretical uncertainty of the BHLUMI calculation of small angle Bhabha scattering has been studied extensively for the event selections of the LEP experiments. The fractional theoretical error is 6.1×10^{-4} for the integrated cross section at LEP1 [13, 18] which was relevant for the determination of the luminosity. Alternative existing calculations have been widely cross-checked with BHLUMI [13]. Moreover the extensive comparisons between data and the predictions of many features predicted by the Monte Carlo by the four LEP collaborations decrease the chances that it contains significant residual technical imperfections. Therefore the estimate of the theoretical uncertainty of BHLUMI is considered solid.

We also used two other Monte Carlo generators which are included in the same BHLUMI package. OLDBIS [20, 21] is an exact $\mathcal{O}(\alpha)$ calculation, based on a MC program written by an independent group. LUMLOG [15, 22] implements a Leading-Log calculation up to $\mathcal{O}(\alpha^3 L^3)$, based on a structure function approach, assuming purely collinear radiation. The BHLUMI package gives access to many intermediate weights which compose the final calculations, so that we could also check several different approximations.

For the purpose of assessing the theoretical uncertainties, our experimental selection has been described by a slightly modified version of the idealized model of the OPAL detector, which is contained in the BHLUMI package (subroutine `TRIOSIW`). This code was also used for the work of [13]. Events were generated within a safely enlarged angular region to protect against loss of visible events. Smearing effects are neglected and an ideal beam geometry is assumed. Nearby particles are combined by a clustering algorithm which has a window matched to the experimental resolution. The energy is defined by summing all the particles inside the isolation cuts in each calorimeter. The position variables R and ϕ are defined as the coordinates of the highest energy particle reconstructed on each side. We applied all the isolation cuts listed in Section 4 to these reconstructed variables. The size of the window used by the clustering algorithm (in R and ϕ) has been varied over a large range to verify the stability of the result. As further checks we used alternative selections, for example, following the nomencalture of [13]), SICAL, which mostly differs in its lack of acollinearity cuts, and BARE, a non-calorimetric selection.

The differential cross section obtained at different perturbative orders is shown in Fig. 10 normalized to the reference BHLUMI cross section. Here vacuum polarization, Z-exchange interference and s-channel photon interference have been switched off. The Born cross section is reduced by about 5-15 % by radiative corrections, depending on the polar angle. The cross section at $\mathcal{O}(\alpha)$ is slightly lower than the reference but in general agrees within 1 %. The exponentiated $\mathcal{O}(\alpha)$ calculation is almost identical to the reference.

Examination of the canonical coefficients [23] indicates that for a calorimetric detector acceptance such as ours the $\mathcal{O}(\alpha^2 L)$ and $\mathcal{O}(\alpha^3 L^3)$ terms dominate the small portion of the complete small-angle Bhahba scattering cross section which is only approximately calculated in BHLUMI. The effect of $\mathcal{O}(\alpha^3 L^3)$ terms on our slope b can be directly calculated using LUMLOG and is found to be small for our selection, $(-7 \pm 13) \times 10^{-5}$.

Although exponentiation allows BHLUMI to include part of the $\mathcal{O}(\alpha^2 L)$ contributions, we choose to assign the entire effect of these terms to the theoretical uncertainty. A rough estimate of this contribution can be obtained from the product of the leading $\mathcal{O}(\alpha L)$ and the subleading $\mathcal{O}(\alpha)$ terms, which is about 0.5×10^{-3} times the Born cross section when integrated over the entire acceptance. To estimate the effect on the running slope, we calculate this product for the inner and outer halves of the acceptance separately and take the difference. The effect on the slope b is then 40×10^{-5}, where a safety factor 2 has been applied.

A better estimate of the $\mathcal{O}(\alpha^2 L)$ contributions, which also includes contributions due to the limited technical precision of the Monte Carlo, can be obtained by comparing the reference $\mathcal{O}(\alpha^2 L^2)$ exponentiated calculation of BHLUMI to the combination of the two independent Monte Carlos, OLDBIS and LUMLOG. We formed the combination used in [13], adding the exact $\mathcal{O}(\alpha)$ given by OLDBIS to the higher orders ($\mathcal{O}(\alpha^2 L^2)$ and $\mathcal{O}(\alpha^3 L^3)$) given by LUMLOG. This result is termed OLDBIS+LUMLOG in what follows, and is compared with several other precise calculations at our disposal in Fig. 11. To quantify possible deviations from the expected t shape, we fit the ratios of alternative calculations to the reference BHLUMI with a linear t dependence. The effect, δb, on our measured slope b is then estimated by accounting for the factor $|t_2 - t_1| / \ln(t_2/t_1) = 3.52$ which converts from linear to logarithmic t range. We obtain:

$$\begin{aligned}
\delta b\,[(\mathrm{OLDBIS} + \mathrm{LUMLOG})/\mathrm{BHLUMI}] &= -20 \pm 31 \times 10^{-5} \\
\delta b\,[\mathrm{Exp.}\mathcal{O}(\alpha)/\mathrm{BHLUMI}] &= +22 \pm 16 \times 10^{-5} \\
\delta b\,[\mathcal{O}(\alpha^2 \mathrm{L}^2)/\mathrm{BHLUMI}] &= -11 \pm 61 \times 10^{-5}
\end{aligned}$$

The ratio of the exponentiated $\mathcal{O}(\alpha)$ calculation to the full BHLUMI is quite flat as a function of t, with a clear normalization shift of 0.2%, which is however irrelevant to our analysis. The unexponentiated $\mathcal{O}(\alpha^2 L^2)$ seems flat too, albeit with much larger statistical error bars. The OLDBIS+LUMLOG combination is rather flat with somewhat larger than expected fluctuations. The extreme points on either ends show downward deviations with respect to the reference BHLUMI, but removing them does not change the fit significantly. We take the statistical error of the (OLDBIS+LUMLOG)/BHLUMI fit, conservatively increased to achieve a good χ^2, as the uncertainty due to missing higher orders, which are mainly $\mathcal{O}(\alpha^2 L)$, plus the technical precision of the calculations. This amounts to 41×10^{-5}, which is coincidentally in exact agrement with our cruder estimate obtained from the product of the leading $\mathcal{O}(\alpha L)$ and the subleading $\mathcal{O}(\alpha)$ terms, above. This estimate is also seen to be in line with the differences observed for the exponentiated $\mathcal{O}(\alpha)$ and $\mathcal{O}(\alpha^2 \mathrm{L}^2)$ calculations. Changing the size of the window used by the cluster algorithm also produces negligible variations. Similar results are also obtained for the SICAL selection.

The interference with the Z-exchange amplitude in the s-channel is a small correction, designated δ_Z in Equation 2, which is not factorized with respect to the main contribution and the running coupling constant. It is energy dependent, vanishing at $\sqrt{s} = m_{\mathrm{Z}}$ and changing sign across the Z pole. In BHLUMI it is calculated up to exact $\mathcal{O}(\alpha)$ photonic corrections, which also can be exponentiated [24]. Vacuum polarization can also be included. Event samples have been generated at three different energies: the Z-peak energy ($\sqrt{s} = 91.1$ GeV) and energies offset by ± 2 GeV. At each energy we consider independently the shifts in the slope b produced by switching off the exponentiation or the vacuum polarization for the calculated interference term δ_Z, and then add these shifts in quadrature. The maximum value, 30×10^{-5}, is taken as the uncertainty due to Z interference.

Concerning the contribution of the vacuum polarization to the δ_Z term, this subtle effect could in principle perturb the asserted cleanliness of the measurement. However such effect is vanishingly small at peak energy (we get -6×10^{-5} at $\sqrt{s} = 91.1$ GeV) and at off-peak energies has about equal and opposite values below $\pm 20 \times 10^{-5}$. This is therefore harmless.

BHLUMI does not include diagrams with extra light pairs (e^+e^-, $\mu^+\mu^-$). Their contribution was calculated explicitly for the OPAL selection, giving a fractional correction of $(-4.4 \pm 1.4) \times 10^{-4}$ [19] on the integrated cross section. The leading order contribution can be checked with LUMLOG, and gives effects on the slope b below 11×10^{-5} with the OPAL or the SICAL data selections.

The estimated theoretical uncertainties are summarized in Table 5. Their quadratic sum is 52×10^{-5} and will be added to the experimental errors.

9 Results

As explained in Section 6, our final results are based solely on the angular distribution observed in the Right calorimeter, and on radial coordinates corrected using the anchoring procedure described in Section 5. The back-to-back nature of Bhabha events implies that the two sides of the detector do not provide independent statistical information concerning the running of α, and the decision to use only the Right calorimeter is based on the desire to reduce possible unassessed systematic errors. Similarly, the decision to use corrected coordinates has little import, since we have already required consistency between the corrected and uncorrected results. Changing any of these choices would shift our result by no more than 10% of its error.

To obtain the final results the radial distribution is binned as specified in Table 1. The numbers of data and Monte Carlo events in each bin for the largest subsample (94b) are reported in Table 6. Note that here the Monte Carlo assumes $\alpha(t) \equiv \alpha_0$. The bin-by-bin acceptance corrections, obtained from Equations 11-12 by inserting the estimated radial biases at the relevant bin boundaries, are also given.

Our selection contains a small irreducible physics background from the process $e^+e^- \rightarrow \gamma\gamma$, for which we apply a correction. Its cross section within our idealized acceptance is found to be 16.9 pb at 91.1 GeV using a Monte Carlo generator including $O(\alpha^3)$ terms [25]. The correction to the slope b is -18×10^{-5}, practically constant with respect to our range of centre-of-mass energies.

The ratio of data to Monte Carlo is fitted with the logarithmic t-dependence of Equation 7 separately for each dataset and the results are reported in Table 7. Both the dominant statistical errors and the experimental systematic errors, which are determined as described in Section 7, are shown. The small corrections for the irreducible background and Z interference have been