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An Escapade in the Tails of the Gaussian

-R. Kellogg

The Fundamentals behind Averaging

- consider the simplest possible situation
- a parent gaussian of zero mean and unit standard deviation

Let's say we sample this distribution by taking a pair of measurements

Everyone knows:

$$\left\langle \frac{x_1 + x_2}{2} \right\rangle = \langle x_1 - x_2 \rangle = 0$$

$$\sqrt{\left\langle \left(\frac{x_1 + x_2}{2} \right)^2 \right\rangle} = \frac{1}{\sqrt{2}}$$

$$\sqrt{\left\langle (x_1 - x_2)^2 \right\rangle} = \sqrt{2}$$

but (on the basis of a small statistical sampling) not everyone knows…

- That

$$\sqrt{\left\langle \left(\frac{x_1 + x_2}{2} \right)^2 \right\rangle_S} = \frac{1}{\sqrt{2}}$$

where S is ***any subsample***
chosen on the basis of $x_1 - x_2$

even way out in the tails $|x_1 - x_2| > 3\sigma_{1-2}$
where σ_{1-2} is the standard deviation of the difference $\left(\sigma_{1-2} = \sqrt{2}\right)$

- This is the marvelous (and somewhat surprising) foundation which supports the validity of averaging over arbitrarily large discrepancies

∴ So, those who favor rigorous, straight averaging are right…

…. As long as the parent distribution is a pure gaussian.

- I have always known this, and have never argued otherwise

- The problem comes down to a paradox

The (second) Statistical Paradox

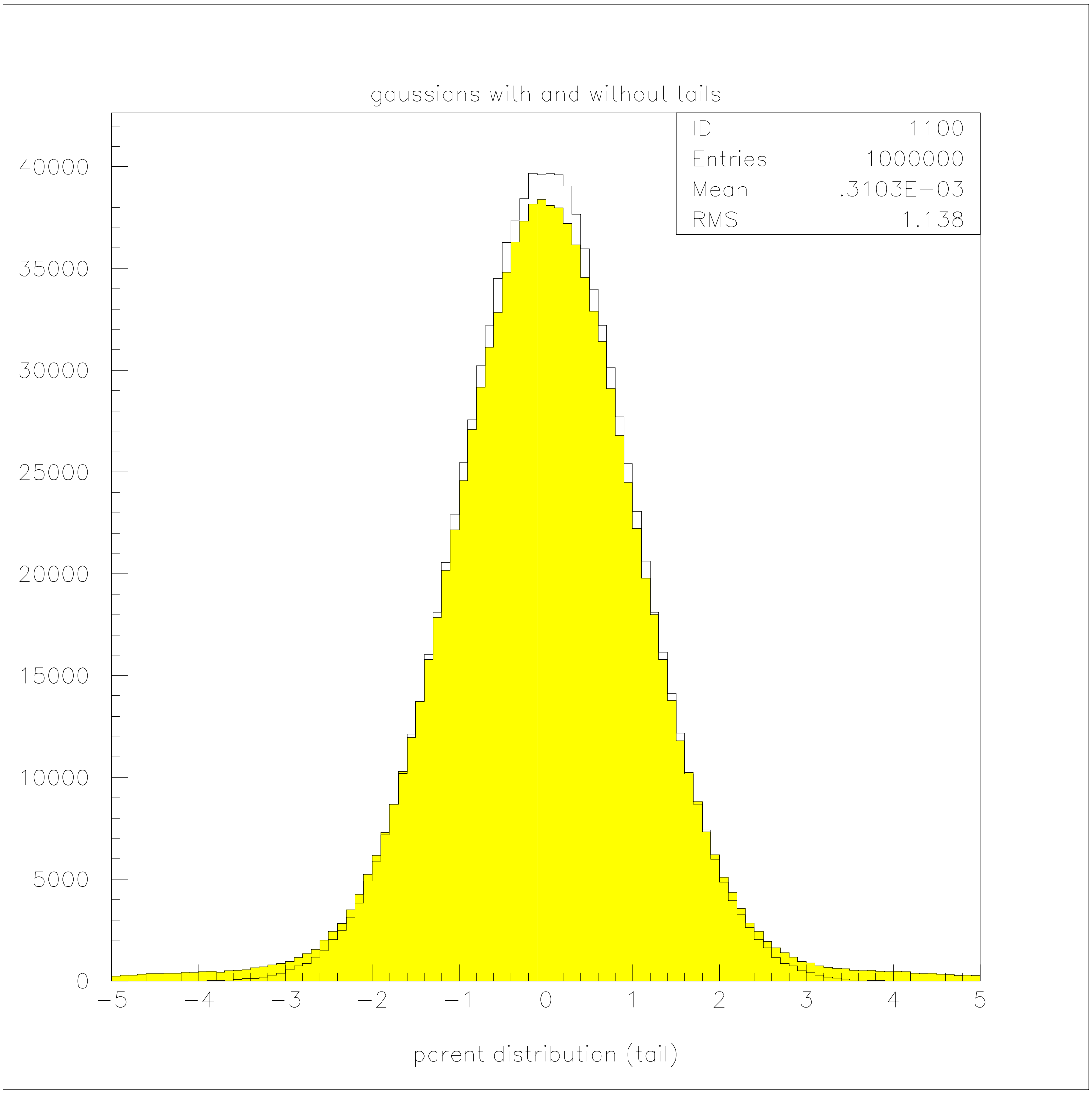
- Everyone knows that all real error distributions are only approximately gaussian

- Everyone treats all errors formally as if they were perfect gaussians

Moving out in the Tail

- Consider a gaussian + a 1% per σ flat tail out to five σ

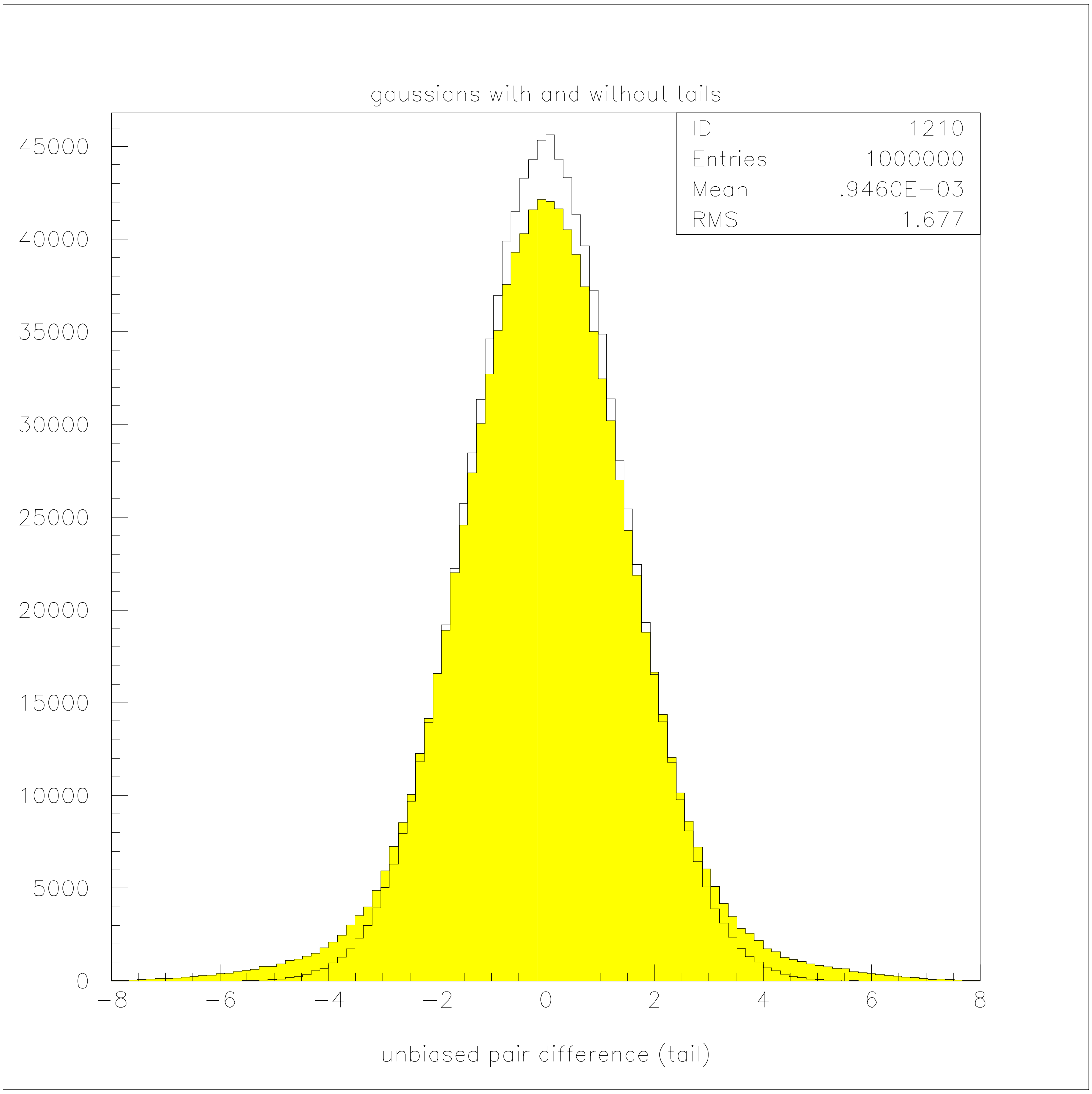
- (you can call this Kellogg's Distribution, if you like)



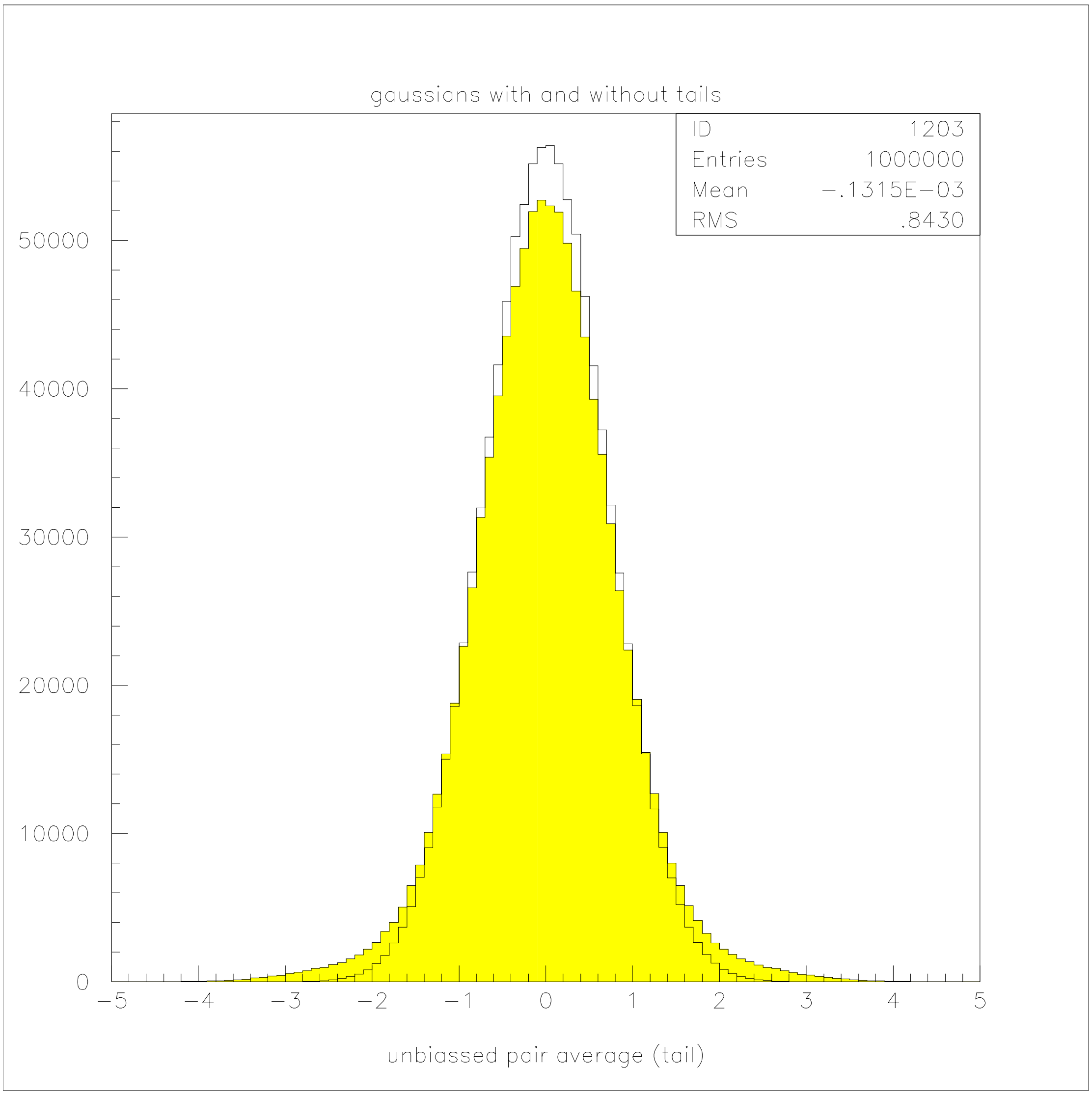
- yes, you can see these tails

- but would a typical error assessment be sensitive to them?

(note, here and in subsequent figures the shaded histogram is Kellogg's Distribution, the unshaded histrogram is a pure gaussian. The statistics correspond to Kellogg's distribution.)



- the distribution of differences $(x_1 - x_2)$ looks pretty gaussian

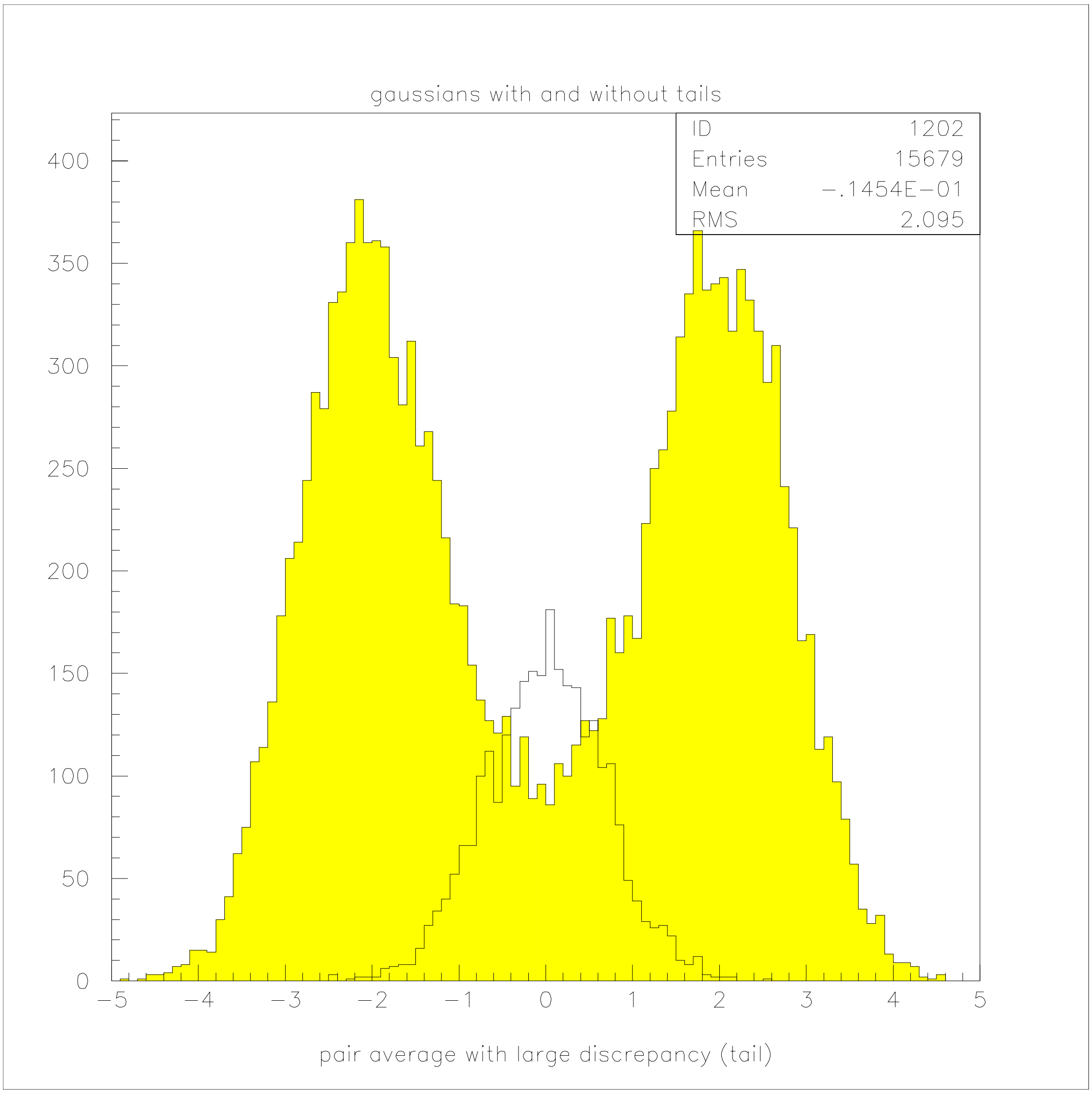


the unbiased distribution of pair-wise averages

$$\frac{x_1 + x_2}{2}$$

also looks pretty gaussian

(good enough for government work, anyway, as we used to say at Brookhaven Lab)



but after requiring $|x_1 - x_2| > 3\sigma_{1-2}$ things start to look a bit sick

Mechanisms

- gaussian tails drop off so fast that “discrepant pairs” are dominated by “one from each wing”

- if the tails drop off more slowly “peak + tail” becomes dominant

<u>Consequences</u>

- Prolog

    - “Kellogg’s Distribution” is purely illustrative

        (any quantitative resemblance to the $\sin^2\theta$ discrepancy is purely intentional)

    - nevertheless…..

it takes a stronger-than-average error analysis to differentiate even much more radical error distributions from a gaussian

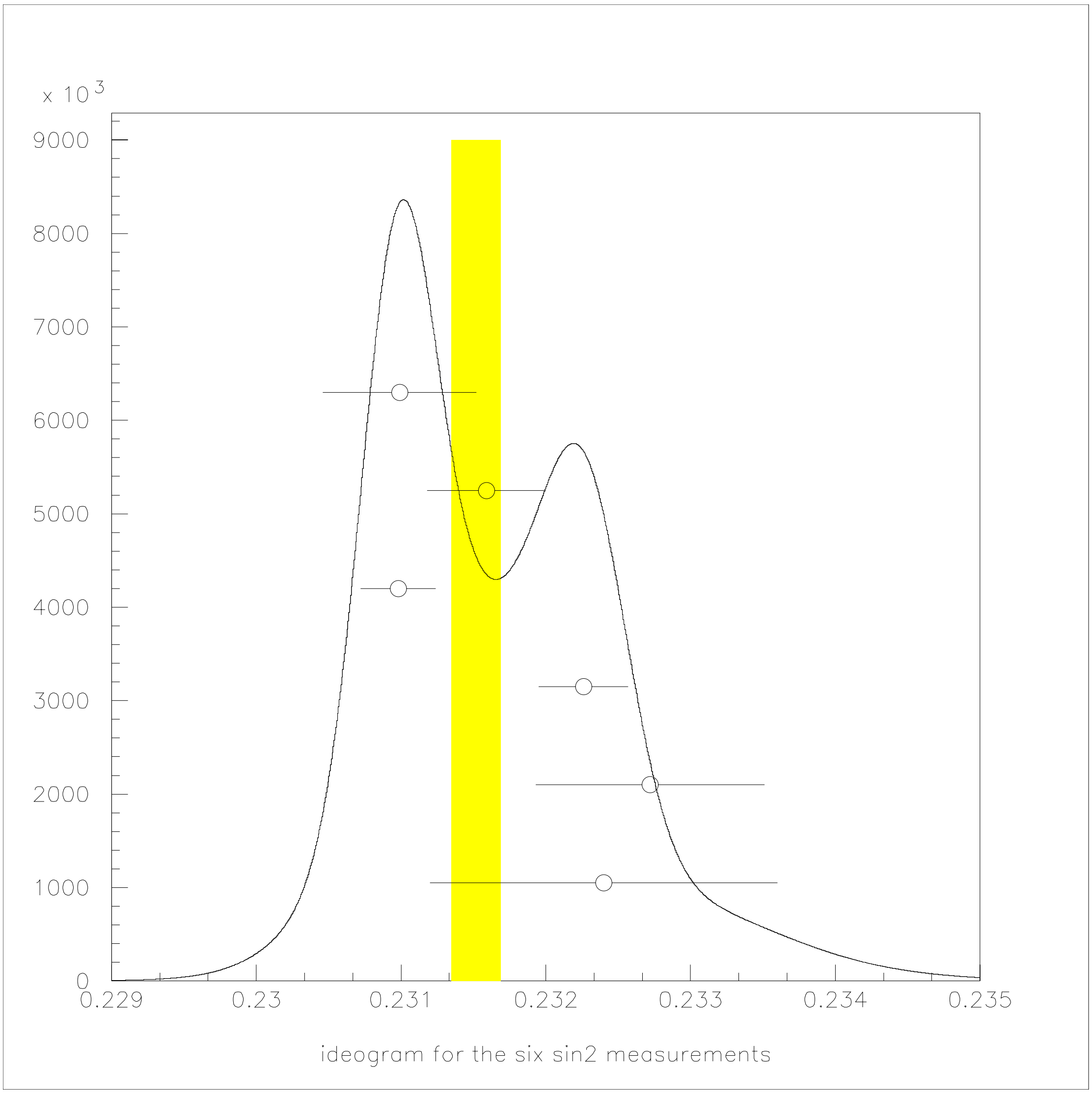
- My Basic Points

    A) The “gaussian myth” is harmless & justified where there is no significant discrepancy

(the core is a safe place for every-day life)

    B) Taking a gaussian average over a significant discrepancy is in fact radical, rather than conservative

(Kellogg is a radical conservative, if you like)



This is an ideogram á la the PDG for the $\sin^2\theta$ measurement summer 2001 (added after the talk)

The shaded band gives the gaussian average